

# SIMPLICIAL DESCENT CATEGORIES

BY

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DISSERTATION

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*To my grandfather Mariano and my grandmother Carmen.*

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## Introduction

In the field of Algebraic Geometry, Grothendieck, at the beginning of the sixties, glimpsed and impelled the introduction of *derived categories* as the appropriate framework to handle the general formulation of duality theorems, either “continuous” –on (quasi)coherent objects– or “discrete” –on the topological analogues motivated by  $\ell$ -adic cohomology– (see for instance the introduction in [Hart]).

Grothendieck’s program culminated in 1963 with Verdier’s thesis, where it is showed up, among other things, the importance of the structure of *triangulated category*. This notion was also related to ideas previously studied by Puppe in the field of Algebraic Topology [V].

Both notions, derived categories and triangulated categories, were fundamental in the important developments in Algebraic Geometry achieved in the period 1960-1975. Nevertheless, at the same time these tools were considered quite sophisticated and not strictly necessary for most questions. Consequently, the use of this theory was not so widespread.

The situation changed dramatically in the last three decades. This was due to several reasons. Among them, the *Riemann-Hilbert correspondence* and the discovering of *perverse sheaves* in Algebraic Geometry. Later, it took place its gradual introduction in Algebraic Topology, Representation Theory, Mathematical Physics and Algebra in general, as well as in -as a feedback- the development of the *theory of motives*.

Nowadays we can see a wide diffusion of both notions, that have become basic tools in Homological Algebra. However, the notion of triangulated category does not seem to be totally satisfactory. For instance, in [GM] the non existence of a functorial cone is remarked (see also [Ne]).

Independently, and also in the sixties, Quillen introduced the notion of *model category* [Q], establishing a general abstract framework to study homotopy categories and even *derived functors*.

On the other hand, at the middle of the twentieth century the notion of *simplicial object* arose to define the singular homology of topological spaces. Since then, (co)simplicial objects have been present in the development of homological and homotopical theories in Algebraic Topology and Algebraic Geometry.

Simplicial sets, and more generally simplicial techniques, are also useful in the framework of model categories, for instance through the natural notion of

*simplicial model category*. Another instance appeared later through the natural action of the homotopy category of simplicial sets on the homotopy category of any model category, which is in fact a key ingredient in the triangulated category structure on the latter one in the stable case (cf. [Ho]).

Model categories are a useful tool in the study of localized categories arising in the *theory of motives* (cf. [FSV], [DLORV]) and more generally in the frameworks of Algebraic Geometry and Homological Algebra, where they are a complement of the notion of triangulated category. Nevertheless, model categories do not always fulfill satisfactorily some common situations. For instance, it is not easy to induce a model category structure on the category of diagrams of a fixed model category. There is also some difficulty in handling filtered structures that often appear in cohomological theories of Algebraic Geometry.

On the other hand, simplicial structures –through the theory of sheaves– have been a relevant tool to deal with certain (multiplicative) constructions, coming from Algebraic Topology, in the framework of Algebraic Geometry [God].

In this work we introduce and develop the notion of *(co)simplicial descent category*, that is an evolution of the corresponding “cubic” notion of Guillén-Navarro [GN]. It is presented as an alternative or complementary instrument to be used in the study of localized (or homotopic) categories arising in Algebraic Geometry.

Let  $\mathcal{D}$  be a category with finite coproducts, endowed with a class of equivalences  $E$ . The category of simplicial objects in  $\mathcal{D}$ ,  $\Delta^\circ \mathcal{D}$ , has an extremely rich structure. Our aim is to transfer this richness to  $\mathcal{D}$  with the help of a “simple” functor  $s : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ . To this end we need to impose some natural compatibility conditions between  $s$  and  $E$ . Thus, a *simplicial descent category* is the data  $(\mathcal{D}, s)$  satisfying certain axioms, as

**Normalization:** the simple of the constant simplicial object associated with  $X$  is equivalent to  $X$

**Exactness:**  $s(\Delta E) \subset E$

**Factorization:** abstraction of Eilenberg-Zilber-Cartier’s theorem appearing in [DP]

**Acyclicity:** the image under  $s$  of the simplicial cone existing in  $\Delta^\circ \mathcal{D}$  must be a cone object in  $\mathcal{D}$  with respect to the equivalence class  $E$ .

Using the simple functor, we obtain *cone* and *cylinder* functors in  $\mathcal{D}$  satisfying the “usual” properties.

The notion of *cosimplicial descent category* turns out to be the dual notion, that is, the opposite category of a simplicial descent category.

## Background

To use a “simple” functor in order to transfer a structure is not a new idea, and it has appeared since the beginning of Topology, for instance when  $\mathbf{s}$  =geometric realization, fact that is emphasized in [May] § 11.

Grothendieck and his school introduced “geometric” simplicial methods in Algebraic Geometry through the so-called *simplicial hypercovers*. They are an essential tool used by Deligne to define a *mixed Hodge structure* on the cohomology of any complex algebraic variety  $S$  (not necessarily smooth).

Technically, the key point is the existence of a suitable “simple” functor

$$\left\{ \begin{array}{c} \text{Cosimplicial Mixed} \\ \text{Hodge Complexes} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Mixed Hodge} \\ \text{complexes} \end{array} \right\}$$

that induces a mixed Hodge structure on the cohomology of  $S$  through a simplicial hypercover  $X$  of  $S$ .

A similar procedure is followed in [DB] to construct a filtered De Rham complex on the cohomology of a singular variety.

Another instance of simple functor motivating this work appears in [N]. Let **Adgc** be the category of commutative differential graded algebras over a field of characteristic 0. In loc. cit. a “simple” functor

$$\{\text{Cosimplicial objects in Adgc}\} \rightarrow \{\text{Adgc}\}$$

is introduced. This functor is known as the *Thom-Whitney simple*. It is used mainly to transfer Sullivan techniques from Algebraic Topology to Algebraic Geometry. As an application, the author endows the rational homotopy spaces of an algebraic variety with a mixed Hodge structure.

In [GN], F. Guillén and V. Navarro Aznar give an axiomatic and abstract notion of “simple” functor, inspired in Deligne’s and Thom-Whitney simples, but formulated in the “cubical” framework [GNPP]. To this end, they introduce the notion of (cubical) *cohomological descent category* and develop an extension

criterion of a functor  $F : \{\text{smooth schemes}\} \rightarrow \mathcal{D}$  to the category of non smooth schemes, provided that  $\mathcal{D}$  is the localized category of a (cubical) cohomological descent category.

In this work we develop the notion of (co)simplicial descent category, that is widely based in the previous notion of (co)homological descent category, where the basic objects are diagrams of cubical shape instead of simplicial objects.

Both notions share the same philosophy, but there are important differences between them:

- cubical diagrams are finite whereas simplicial objects are infinite.

- in the cubical case, the factorization axiom is not so strong, in fact it is usually an automatic consequence of the Fubini theorem on the index swapping in a double coend. However, the simplicial factorization axiom is much stronger, because it involves the diagonal object associated with a bisimplicial object, and it is in fact an abstraction of the Eilenberg-Zilber-Cartier theorem given in [DP]. This theorem uses the degeneracy maps of simplicial objects. Hence, strict simplicial objects (with no degeneracy maps) are not enough for our purposes. Nevertheless, a cubical diagram does not have degeneracy maps.

On the other hand, working in the simplicial framework has other advantages. For instance, one can induce a natural action of  $\Delta^\circ \text{Set}$  on  $\mathcal{D}$  (defined through the simple functor from the action of  $\Delta^\circ \text{Set}$  on  $\Delta^\circ \mathcal{D}$ ).

We can also exploit the homotopy structure of  $\Delta^\circ \mathcal{D}$  when  $\mathcal{D}$  is a simplicial descent category. It turns out that homotopic morphisms between simplicial objects (in the classical sense of simplicial homotopy) are mapped by  $\mathbf{s}$  into identical morphisms in the localized category  $\mathcal{D}[\mathbf{E}^{-1}]$ . In particular, (simplicial) homotopy equivalencies are mapped by  $\mathbf{s}$  into equivalences. This applies to augmentations with an “extra degeneracy”.

## Main results

**a)** We establish a set of axioms for the *(co)simplicial descent categories*. These axioms unify the properties satisfied by a significant number of examples in the frameworks of Algebraic Topology and Algebraic Geometry.

To be precise, a category  $\mathcal{D}$  with finite coproducts and final object  $*$  together with a saturated class of equivalences  $\mathbf{E}$  and a “simple” functor  $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  is a *simplicial descent category* if the following axioms are satisfied.

**Additivity:** The canonical morphism  $\mathbf{s}X \sqcup \mathbf{s}Y \rightarrow \mathbf{s}(X \sqcup Y)$  is an equivalence



for any  $X, Y \in \Delta^\circ \mathcal{D}$ . Also  $E \sqcup E \subseteq E$ .

**Factorization:** There exists a natural equivalence  $\mu_Z : \mathbf{s}DZ \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}Z$ , where  $DZ$  is the diagonal object associated with the bisimplicial object  $Z$  and  $\mathbf{s}\Delta^\circ \mathbf{s}Z$  is its iterated simple.

**Normalization:** There exists a natural equivalence  $\lambda_X : \mathbf{s}(X \times \Delta) \rightarrow X$ , compatible with  $\mu$ , relating an object  $X$  in  $\mathcal{D}$  to the simple of its associated constant simplicial object  $X \times \Delta$ .

**Exactness:** If  $f$  is a morphism in  $\Delta^\circ \mathcal{D}$  such that  $f_n \in E$  for all  $n$  then  $\mathbf{s}f \in E$ .

**Acyclicity:** The simple of a morphism  $f$  in  $\Delta^\circ \mathcal{D}$  is an equivalence if and only if the simple of its simplicial cone,  $\mathbf{s}Cf$ , is acyclic.

**Symmetry:** The class  $\{f : X \rightarrow Y \mid \mathbf{s}f \in E\}$  is invariant by the operation of inverting the order of the face and degeneracy maps of  $X$  and  $Y$ .

**b)** We introduce the following ‘transfer lemma’, that will be widely used to produce examples of (co)simplicial descent categories

*Assume that  $(\mathcal{D}', \mathbf{s}', E', \lambda', \mu')$  is a simplicial descent category and  $\mathcal{D}$  is a category with a simple functor  $\mathbf{s}$  and “compatible” natural transformations  $\lambda$  and  $\mu$ . Moreover, let  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  be a functor such that the following diagram*

$$\begin{array}{ccc} \Delta^\circ \mathcal{D} & \xrightarrow{\psi} & \Delta^\circ \mathcal{D}' \\ \mathbf{s} \downarrow & & \downarrow \mathbf{s}' \\ \mathcal{D} & \xrightarrow{\psi} & \mathcal{D}' \end{array}$$

*commutes up to natural equivalence, “compatible” with transformations  $\lambda$ ,  $\lambda'$  and  $\mu$ ,  $\mu'$ . Then  $(\mathcal{D}, E = \psi^{-1}E')$  is also a simplicial descent category.*

**c)** In any simplicial descent category  $\mathcal{D}$  we can induce cone and cylinder functors in the following way.

Given a morphism  $f : X \rightarrow Y$  in  $\mathcal{D}$ , consider it in  $\Delta^\circ \mathcal{D}$  as a constant simplicial map  $f \times \Delta : X \times \Delta \rightarrow Y \times \Delta$ . Then, the cone of  $f$  is by definition  $\mathbf{s}C(f \times \Delta)$ , where  $C$  is the simplicial cone associated with  $f \times \Delta$ .

Similarly, the “cylinder” of two morphisms  $A \xleftarrow{f} B \xrightarrow{g} C$  in  $\mathcal{D}$  is  $\mathbf{s}Cyl(f \times \Delta, g \times \Delta)$ , where  $Cyl$  is the simplicial cylinder associated with  $(f \times \Delta, g \times \Delta)$ .

When  $X$  is a simplicial set, then the classical cylinder associated with  $X$  is just our cylinder of  $X = X = X$ , and the one associated with a morphism  $f$  is our cylinder of  $X = X \xrightarrow{f} Y$ .

**d)** We provide a “reasonable” description of the morphisms in  $Ho\mathcal{D} = \mathcal{D}[E^{-1}]$ , the homotopy category of  $\mathcal{D}$ . In general the class of equivalences  $E$  does not has calculus of fractions. The key point to obtain this description is the cylinder functor and its properties.

More specifically, consider the functor  $R : \mathcal{D} \rightarrow \mathcal{D}$  defined by  $RX := \mathbf{s}(X \times \Delta)$ . Note that the natural transformation  $\lambda : R \rightarrow Id$  is a pointwise equivalence. Then a morphism  $f : X \rightarrow Y$  in  $Ho\mathcal{D}$  is represented by a sequence

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f'} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y .$$

where all arrows except  $f'$  are equivalences. Using this description we can prove, for instance, that  $Ho\mathcal{D}$  is additive when  $\mathcal{D}$  is so.

e) The shift  $[1] : \mathcal{D} \rightarrow \mathcal{D}$  is defined by  $X[1] = c(X \rightarrow *)$ , where  $c$  is the cone functor given above. We consider the class of distinguished triangles in  $Ho\mathcal{D}$  consisting of those triangles isomorphic, for some  $f$ , to

$$X \xrightarrow{f} Y \longrightarrow c(f) \longrightarrow X[1] .$$

Distinguished triangles satisfy all axioms of triangulated category except the second axiom TR 2 (that is, the one involving the shift of distinguished triangles), with no extra assumptions (neither additivity).

Moreover, in the additive case  $Ho\mathcal{D}$  is a “suspended” (or right triangulated) category [KV] in the simplicial case, and it is “cosuspended” (or left triangulated) in the cosimplicial case. In particular, if in addition the shift is an automorphism of  $Ho\mathcal{D}$  then  $Ho\mathcal{D}$  is a triangulated category.

f) In order to study the properties of the simplicial cone and cylinder functors, we develop a much more general construction, the “*total simplicial object*” associated with a “biaugmented bisimplicial object” (or, more generally, to a “ $n$ -augmented  $n$ -simplicial object”). More concretely, consider the bisimplicial object  $Z$  given by the picture

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \dots \\
Z_{-1,2} & \longleftarrow & Z_{0,2} & \rightleftarrows & Z_{1,2} & \rightleftarrows & Z_{2,2} & \rightleftarrows & Z_{3,2} & & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \dots \\
Z_{-1,1} & \longleftarrow & Z_{0,1} & \rightleftarrows & Z_{1,1} & \rightleftarrows & Z_{2,1} & \rightleftarrows & Z_{3,1} & & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \dots \\
Z_{-1,0} & \longleftarrow & Z_{0,0} & \rightleftarrows & Z_{1,0} & \rightleftarrows & Z_{2,0} & \rightleftarrows & Z_{3,0} & & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \dots \\
& & Z_{0,-1} & \rightleftarrows & Z_{1,-1} & \rightleftarrows & Z_{2,-1} & \rightleftarrows & Z_{3,-1} & & \dots
\end{array}$$

Then the total simplicial object associated with a biaugmented is in degree  $k$  the coproduct of the  $k$ -th diagonal of  $Z$ . The face and degeneracy maps are defined respectively as coproducts of those of  $Z$ .

It turns out that this total functor is left adjoint to the “total decalage” given in [III] p.7. We can also consider the total functor as the simplicial analogue to the total chain complex associated with a double complex.

**g)** On one hand, we have checked that all examples of (cubical) (co)homological descent categories [GN] are simplicial descent categories. Among them we can mention (filtered) cochain complexes, topological spaces or commutative differential graded algebras.

On the other hand, we provide other examples than the cubical ones. For instance, we consider a category of mixed Hodge complexes, and we endow it with a cosimplicial descent category structure. In this structure the simple functor is just the one developed in [DeIII]. As a corollary we obtain a triangulated structure on its homotopy category (similar to the one obtained in [Be]).

Also the category of DG-modules over a fixed DG-category is a cosimplicial descent category, and we deduce the usual triangulated structure existing in its homotopy category [K].

A possibly less known example is the category of cochain complexes together with a biregular filtration, where the class  $E_2$  of equivalences consists of those morphisms which induce isomorphism in the second term of the respective spectral sequences.

Now we get a triangulated structure on the category of bounded-below filtered complexes localized with respect to the class  $E_2$ . Moreover, the “decalage” functor of a filtration [DeII] I.3.3 is a triangulated functor with values in the (usual) filtered derived category.

## Contents

CHAPTER 1: The first chapter contains the simplicial/combinatorial preliminaries.

We study the classical cone and cylinder functors in  $\Delta^\circ \mathcal{D}$  as particular cases of the *total functor* of biaugmented bisimplicial objects, that was mentioned before. As far as the author knows, this total functor has not been previously studied. Particular and related cases can be found in [EP] and [AM]).

The total functor satisfies interesting properties. For instance, the iteration of totals of  $n$ -augmented  $n$ -simplicial objects does not depend on the order in which we compute it (analogously to the property of the total complex associ-

ated with a multiple chain complex).

CHAPTER 2: This chapter contains the definition of (co)simplicial descent category, as well as some of their properties, mainly those of homotopical type, related to the cone and cylinder functors. The axioms of (co)simplicial descent category are natural in the following sense: If  $I$  is a small category and  $\mathcal{D}$  is a (co)simplicial descent category then the category of functors from  $I$  to  $\mathcal{D}$  (endowed with the pointwise simple and the pointwise equivalences) is again a (co)simplicial descent category.

In section 2.3 the “factorization” property of the cylinder is established. In terms of the cone functor this property means the following: Consider a commutative diagram  $C$  in a simplicial descent category  $\mathcal{D}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array} .$$

If we apply the cone functor by rows and columns respectively we get

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \begin{array}{c} c(f) \\ \downarrow \alpha \\ c(f') \end{array}$$

$$c(g) \xrightarrow{\beta} c(g') .$$

Then the cone of  $\alpha$  and the cone of  $\beta$  are equivalent in a natural way. This fact will play an important role in chapter 4, since it is the key point in the proofs of the octahedron axiom and of the second axiom of triangulated categories. Section 2.4 is devoted to the study of the properties of the square

$$\begin{array}{ccc} \mathbf{s}X & \xrightarrow{\mathbf{s}f} & \mathbf{s}Y \\ \mathbf{s}\epsilon \downarrow & & \downarrow I_Y \\ \mathbf{s}X_{-1} & \xrightarrow{I_{X_{-1}}} & \mathbf{s}Cyl(f, \epsilon) \end{array}$$

obtained by applying  $\mathbf{s}$  to the respective square in  $\Delta^\circ \mathcal{D}$  induced by the simplicial cylinder. One can check that this square “commutes up to equivalence”, and that  $I_{X_{-1}}$  is an equivalence provided that  $\mathbf{s}f$  is so. The reciprocal assertion also holds under some extra assumptions, and it will be needed to prove the “transfer lemma”.

In section 2.5 we introduce the notion of functor of simplicial descent categories,

and we prove the “transfer lemma”.

CHAPTER 3: In this chapter we give a “reasonable” description of the morphisms of  $Ho\mathcal{D}$ . We use this description to prove that  $Ho\mathcal{D}$  is additive if  $\mathcal{D}$  is so.

CHAPTER 4: We prove here that the class of distinguished triangles defined through the cone functor

$$X \xrightarrow{f} Y \longrightarrow c(f) \longrightarrow X[1]$$

satisfies the axioms TR1, TR3 and TR4 of triangulated categories (with no extra assumptions). In the additive case, the right implication of TR2 holds, so  $\mathcal{D}$  is a “suspended” category [KV]. Moreover, if the shift is an isomorphism of categories then  $Ho\mathcal{D}$  is a triangulated category.

CHAPTER 5: In this chapter we exhibit examples of simplicial descent categories. The first one is the category of chain complexes in an additive or abelian category, taking as equivalences  $E$  =homotopy equivalences or  $E$  =quasi-isomorphisms. In this example the axioms of simplicial descent category are checked “by hand”, whereas in the remaining simplicial examples they are checked by means of the transfer lemma.

The following picture contains the main examples of simplicial descent categories included in this chapter as well as the functors of simplicial descent categories between them

$$Top \xrightleftharpoons[|\cdot|]{S} \Delta^\circ Set \longrightarrow \Delta^\circ Ab \longrightarrow Ch_* Ab \ .$$

CHAPTER 6: Examples of cosimplicial descent categories are provided in this chapter. Cochain complexes are obtained just as the dual case of chain complexes. In section 6.2 the category of commutative differential graded algebras (over a field of characteristic 0) is considered. The simple is just the ‘Thom-Whitney simple’ given in [N].

In section 6.3 we endow the category of DG-modules [K] with a cosimplicial descent category structure.

In the next section we prove that the category of (positive) complexes together with a (biregular) filtration,  $CF^+\mathcal{A}$  has two different cosimplicial descent structures. In the first one,  $({}_1CF^+\mathcal{A}, E)$ , the equivalences  $E$  are the filtered quasi-isomorphisms. In the second one,  $({}_2CF^+\mathcal{A}, E_2)$ , the class of equivalences  $E_2$  consists of those morphisms inducing isomorphism in the second term of the

spectral sequence. These statements follow from the transfer lemma. It is applied twice to the functors contained in the following diagram

$${}_2\mathrm{CF}^+ \mathcal{A} \xrightarrow{Dec} {}_1\mathrm{CF}^+ \mathcal{A} \xrightarrow{Gr} (Ch^* \mathcal{A})^{\mathbb{Z}} .$$

The functor  $Dec$  is the decalage functor given in [DeII] I.3.3, and  $Gr$  is the graded functor, with values in the category of  $\mathbb{Z}$ -graded cochain complexes, endowed with the degreewise descent category structure.

Both structures are used to induce a cosimplicial descent category structure on “the” category of mixed Hodge complexes.

To finish, we include an appendix containing the Eilenberg-Zilber-Cartier theorem [DP], and some extra properties which are not easy to find in the existing literature.

### Further research/Open problems

Next we list some questions and problems related to this work. Some of them are natural questions and others are further applications and complements.

**I.** The category  $Op(\mathcal{D})$  of operads over a symmetric monoidal descent category  $\mathcal{D}$  has a natural structure of descent category (joint work with A. Roig). A related open problem is to endow the category of operadic algebras over a fixed operad  $\mathcal{P} \in Op(\mathcal{D})$  with a structure of cosimplicial descent category.

**II.** Every cubical diagram  $X$  in a fixed category  $\mathcal{D}$  gives rise in a natural way to a simplicial object in  $\mathcal{D}$ ,  $\tau X$  ([N], 12.1). If  $\mathcal{D}$  is a simplicial descent category, we can compose  $\tau$  with the simple functor  $s : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ , obtaining in this way a “cubical simple functor”  $\{\text{Cubical diagrams in } \mathcal{D}\} \rightarrow \mathcal{D}$ . The following natural question arises:

Is every (co)simplicial descent category a “cubical” (co)homological descent category in the sense of [GN]? If the answer is affirmative, then the “extension of functors” theorem, given in –loc. cit.–, will be also valid for functors with values in the localized category of a cosimplicial descent category.

**III.** The localized category  $Ho\mathcal{D}$  of a descent category  $\mathcal{D}$  has a translation functor  $T : Ho\mathcal{D} \rightarrow Ho\mathcal{D}$  as well as a class of distinguished triangles (coming from the cone functor in  $\mathcal{D}$ ).

The following question is motivated by model category theory:

Is  $Ho\mathcal{D}$  an additive category provided that  $T$  is an automorphism in  $Ho\mathcal{D}$  (stable case)?

If the translation functor  $T : Ho\mathcal{D} \rightarrow Ho\mathcal{D}$  is not an automorphism, it would

be interesting to provide an abstract process of ‘stabilization’, similar to the construction of the category of spectra from the category of topological spaces.

**IV.** Study the properties of the action of  $\Delta^\circ Set$  over a descent category  $\mathcal{D}$  inherited from the natural action of  $\Delta^\circ Set$  on  $\Delta^\circ \mathcal{D}$  through the simple functor. Furthermore, possibility of carrying this action to the level of derived categories. That is, to check whether this action gives rise to another one from  $Ho\Delta^\circ Set$  on  $Ho\mathcal{D}$  or not.

**V.** Define sheaf cohomology with values in a descent category  $\mathcal{D}$ , in a similar way to that given in [N] for the case  $\mathcal{D}$ =commutative differential graded algebras. In a more general sense, to tackle the definition of derived functors in descent categories, following the recent work [GNPR].

**VI.** Relationship between simplicial descent categories and model categories. In the “cubical” case, it is known that the subcategory of fibrant objects in a simplicial model category is a cohomological descent category [R].

**VII.** Extension of the notion of descent category to the context of fibred categories and stacks.

**VIII.** Study the relationship with the recent work [Vo], where it is also used the simplicial cylinder  $\widetilde{Cyl}$  introduced in the first chapter of this work. For instance, if  $\mathcal{D}$  is a simplicial descent category, it holds that the class  $\{f \mid sf \in E\}$  is a  $\Delta$ -closed class in the sense of loc. cit..

# Chapter 1

## Preliminaries

### 1.1 Simplicial objects

In this section we will remind the definition and some basic properties of simplicial objects in a fixed category  $\mathcal{C}$ . For a more detailed exposition see [May], [GZ] or [GJ].

**DEFINITION 1.1.1** (The Simplicial Category  $\Delta$ ).

The simplicial category  $\Delta$  has as objects the ordered sets  $[n] \equiv \{0, \dots, n\}$  with  $0 < 1 < \dots < n$ , and as morphisms the (weak) monotone functions, that is

$$\text{Hom}_\Delta([m], [n]) = \{f : [m] \longrightarrow [n] \text{ with } f(i) \leq f(j) \text{ if } i \leq j\}.$$

There exists two kind of relevant morphisms in the category  $\Delta$ .

The face morphisms  $\partial_i = \partial_i^n : [n-1] \rightarrow [n]$  are just those monotone functions such that  $\partial_i(\{0, \dots, n-1\}) = \{0, \dots, i-1, i+1, \dots, n\}$ , for all  $i = 0, \dots, n$ .

The degeneracy morphisms  $\sigma_i^n = \sigma_i : [n+1] \rightarrow [n]$  are characterized by  $\sigma_i(i) = \sigma_i(i+1) = i$ , for all  $i = 0, \dots, n$ .

More specifically,  $\partial_i(l) = l$  if  $l \leq i-1$  and  $\partial_i(l) = l+1$  if  $l \geq i$ , whereas  $\sigma_i(l) = l$  if  $l \leq i$  and  $\sigma_i(l) = l-1$  if  $l > i$ .

These morphisms satisfy the so called “simplicial identities”, that are the following equalities

$$\begin{aligned} \partial_j^{n+1} \partial_i^n &= \partial_i^{n+1} \partial_{j-1}^n & \text{if } i < j \\ \sigma_j^n \sigma_i^{n+1} &= \sigma_i^n \sigma_{j+1}^{n+1} & \text{if } i \leq j \\ \sigma_j^{n-1} \partial_i^n &= \begin{cases} \partial_i^n \sigma_{j-1}^{n-2} & \text{if } i < j \\ Id_{[n-1]} & \text{if } i = j \text{ or } i = j+1 \\ \partial_{i-1}^{n-1} \sigma_j^{n-2} & \text{if } i > j+1. \end{cases} \end{aligned} \tag{1.1}$$



The category  $\Delta$  is generated by the face and degeneracy morphisms, as described in the following proposition [May].

**PROPOSITION 1.1.2.** *Let  $f : [n] \rightarrow [m]$  be a morphism in  $\Delta$  different from the identity. Denote by  $i_1 > i_2 \cdots > i_s$  those elements of  $[m]$  that do not belong to the image of  $f$ , and by  $j_1 < j_2 \cdots < j_l$  those elements of  $[n]$  such that  $f(j_k) = f(j_k + 1)$ . Then*

$$f = \partial_{i_1} \cdots \partial_{i_s} \sigma_{j_1} \cdots \sigma_{j_l}, \quad (1.2)$$

Moreover, the factorization of  $f$  in this way is unique.

**REMARK 1.1.3.** Let  $\widehat{\Delta}$  be the category whose object are all the (non empty) finite ordered sets, and whose morphisms are the monotone maps.

If  $E = \{e_0 < e_1 < \cdots < e_n\}$  is an object of  $\widehat{\Delta}$ , then  $E$  is canonically isomorphic to  $[n]$ , and  $n$  is the cardinal of  $E$  minus 1.

Then each object of  $\widehat{\Delta}$  is isomorphic to a unique object of  $\Delta$ , or equivalently,  $\Delta$  is a skeletal subcategory of  $\widehat{\Delta}$ . Then it follows the existence of a functor  $p : \Delta \rightarrow \widehat{\Delta}$  quasi-inverse of the inclusion  $i : \Delta \rightarrow \widehat{\Delta}$ .

The intrinsic meaning of some definitions and properties given in terms of  $\Delta$  become clarified when expressed in terms of  $\widehat{\Delta}$ , as we will check along this chapter.

We will use the following operations relative to  $\widehat{\Delta}$ .

(1.1.4) Denote by

$$+ : \widehat{\Delta} \times \widehat{\Delta} \rightarrow \widehat{\Delta}$$

the “ordered sum” of ordered sets.

That is, if  $E$  and  $F$  are objects of  $\widehat{\Delta}$ , then  $E + F$  is  $E \sqcup F$  as a set. The order in  $E + F$  is the one compatible with those of  $E$  and  $F$ , and such that  $e < f$  if  $e \in E$  and  $f \in F$ .

Analogously,  $+ : \Delta \times \Delta \rightarrow \Delta$  is such that  $[n] + [m] = p(i([n]) + i([m])) = [n + m + 1]$ , where  $[n]$  is identified with  $\{0, \dots, n\} \subset [n + m + 1]$  and  $[m]$  with  $\{n + 1, \dots, n + m + 1\} \subset [n + m + 1]$ .

(1.1.5) Denote by

$$\widehat{op} : \widehat{\Delta} \rightarrow \widehat{\Delta}$$

the functor which consists of taking the opposite order. That is,  $op(E)$  is equal to  $E$  as a set, but it has the inverse order of  $E$ .

Analogously,  $op = p \circ \widehat{op} \circ i : \Delta \rightarrow \Delta$  is the functor given by

$$op([n]) = [n] \text{ and if } \theta : [n] \rightarrow [m], (op(\theta))(i) = m - \theta(n - i),$$

and then

$$op(\partial_i^n) = \partial_{n-i}^n : [n-1] \rightarrow [n] \text{ and } op(\sigma_j^n) = \sigma_{n-j}^n : [n+1] \rightarrow [n].$$

**DEFINITION 1.1.6** (The Category  $\Delta_e$ ).

Let  $\Delta_e$  be the strict simplicial category, that is the subcategory of  $\Delta$  with the same objects, but whose morphisms are the injective monotone functions.

Analogously,  $\Delta_e$  is generated by the face morphisms, and it is a skeletal subcategory of the corresponding category  $\widehat{\Delta}_e$ .

**DEFINITION 1.1.7** (Simplicial Objects).

A simplicial object  $X$  in a category  $\mathcal{C}$  is a contravariant functor from the simplicial category to  $\mathcal{C}$ , that is,  $X : \Delta^\circ \rightarrow \mathcal{C}$ .

(1.1.8) As a corollary of (1.2),  $X$  is characterized by the data

$$X_p = X([p]) \quad d_i = X(\partial_i) \quad s_j = X(\sigma_j)$$

where the face and degeneracy maps  $d_i$  and  $s_j$  of  $X$  satisfy the following equalities, also called simplicial identities :

$$\begin{aligned} d_i^n d_j^{n+1} &= d_{j-1}^n d_i^{n+1} \text{ if } i < j \\ s_i^{n+1} s_j^n &= s_{j+1}^{n+1} s_i^n \text{ if } i \leq j \\ d_i^n s_j^{n-1} &= \begin{cases} s_{j-1}^{n-2} d_i^n & \text{if } i < j \\ Id_{[n-1]} & \text{if } i = j \text{ or } i = j + 1 \\ s_j^{n-2} d_{i-1}^{n-1} & \text{if } i > j + 1. \end{cases} \end{aligned} \quad (1.3)$$

Then, a simplicial object  $X = \{X_n, d_i, s_j\}$  can be represented as follows

$$\begin{array}{ccccccc} X_0 & \xleftarrow{d_1} & X_1 & \xleftarrow{\quad} & X_2 & \xleftarrow{\quad} & X_3 & \cdots \\ & \searrow^{s_0} & \swarrow_{d_0} & & \swarrow_{d_1} & \swarrow_{d_2} & \swarrow_{d_3} & \\ & & & & & & & \end{array}$$

(1.1.9) Analogously, a strict simplicial object is  $X : \Delta_e^\circ \rightarrow \mathcal{C}$ , that is given by

$$\begin{array}{ccccccc} X_0 & \xleftarrow{d_1} & X_1 & \xleftarrow{\quad} & X_2 & \xleftarrow{\quad} & X_3 & \cdots \\ & \searrow^{d_0} & & & & & & \end{array}$$

(1.1.10) Dually, a cosimplicial object in  $\mathcal{C}$  is a functor  $X : \Delta \rightarrow \mathcal{C}$ , or equivalently, a simplicial object in  $\mathcal{C}^\circ$ . The strict cosimplicial objects in  $\mathcal{C}$  are defined in the same way.

**REMARK 1.1.11.** From now on, we will use the notation  $\mathcal{IC}$  for the category of functors from  $\mathcal{I}$  to  $\mathcal{C}$ .

**DEFINITION 1.1.12.** The simplicial objects in  $\mathcal{C}$  give rise to the category  $\Delta^\circ\mathcal{C}$ , whose morphisms are the natural transformations between functors.

A morphism  $\rho : X \rightarrow X'$  in  $\Delta^\circ\mathcal{C}$  is a set of morphisms  $\rho_n : X_n \rightarrow X'_n$  in  $\mathcal{C}$  commuting with the face and degeneracy maps, that is

$$\rho_n d_X = d_{X'} \rho_{n+1} \quad \rho_{n+1} s_X = s_{X'} \rho_n .$$

(1.1.13) Having into account 1.1.3,  $\Delta^\circ\mathcal{C}$  is canonically equivalent to  $\widehat{\Delta}^\circ\mathcal{C}$ . Denote by  $\mathcal{I} : \widehat{\Delta}^\circ\mathcal{C} \rightarrow \Delta^\circ\mathcal{C}$  (resp.  $\mathcal{P} : \Delta^\circ\mathcal{C} \rightarrow \widehat{\Delta}^\circ\mathcal{C}$ ) the functor induced by composition with  $i : \Delta \rightarrow \widehat{\Delta}$  (resp.  $p : \widehat{\Delta} \rightarrow \Delta$ ).

Since  $p \circ i = Id_\Delta$ , we have that  $\mathcal{I} \circ \mathcal{P} = Id_{\Delta^\circ\mathcal{C}}$ .

(1.1.14) The categories of cosimplicial, strict simplicial and strict cosimplicial objects in  $\mathcal{C}$  are denoted respectively by  $\Delta\mathcal{C}$ ,  $\Delta_e^\circ\mathcal{C}$  and  $\Delta_e\mathcal{C}$ , and are defined in the same way as  $\Delta^\circ\mathcal{C}$ .

Composing with the inclusion  $\Delta_e \rightarrow \Delta$  we obtain the forgetful functor

$$U : \Delta^\circ\mathcal{C} \rightarrow \Delta_e^\circ\mathcal{C}$$

consisting in forgetting the degeneracy maps of a simplicial object.

**REMARK 1.1.15.** If  $\mathcal{C}$  has colimits, the forgetful functor  $U : \Delta^\circ\mathcal{C} \rightarrow \Delta_e^\circ\mathcal{C}$  has a left adjoint  $\pi$  (called “Dold-Puppe transformation”) constructed as usual. However, it follows from the property 1.1.2 that  $\pi$  exists just assuming the existence of coproducts in  $\mathcal{C}$ . Next we remind the definition of the Dold-Puppe transformation (cf. [G] 1.2).

**PROPOSITION 1.1.16.** *If  $\mathcal{C}$  is a category with finite coproducts, the forgetful functor  $U : \Delta^\circ\mathcal{C} \rightarrow \Delta_e^\circ\mathcal{C}$  admits a left adjoint  $\pi : \Delta_e^\circ\mathcal{C} \rightarrow \Delta^\circ\mathcal{C}$ . If  $A$  is a strict simplicial object in  $\mathcal{C}$ , then  $\pi A$  is defined as*

$$(\pi A)_n = \coprod_{\theta : [n] \rightarrow [m]} A_m^\theta$$

where the coproduct is indexed over the set of surjective morphisms  $\theta : [n] \twoheadrightarrow [m]$  in  $\Delta$  and  $A_m^\theta = A_m$ .

**(1.1.17)** Let us see how the action of  $\pi A$  over the morphisms in  $\Delta$  is defined. Let  $f : [n'] \rightarrow [n]$  be a morphism in  $\Delta$ . The morphism in  $\mathcal{C}$

$$(\pi A)(f) : (\pi A)_n = \coprod_{\theta : [n] \twoheadrightarrow [m]} A_m^\theta \longrightarrow (\pi A)_{n'} = \coprod_{\rho : [n'] \twoheadrightarrow [m']} A_{m'}^\rho$$

is defined as follows.

Given a surjective  $\theta : [n] \twoheadrightarrow [m]$ , it follows from 1.1.2 that there exists a unique factorization of  $\theta \circ f : [n'] \rightarrow [m]$  as

$$[n'] \xrightarrow{\alpha} [l] \xhookrightarrow{\beta} [m]$$

where  $\alpha$  is surjective and  $\beta$  is injective.

Then the restriction of  $(\pi A)(f)$  to  $A_m^\theta$  is just

$$A(\beta) : A_m^\theta \rightarrow A_l^\alpha .$$

**DEFINITION 1.1.18** (Bisimplicial objects).

A bisimplicial object in  $\mathcal{C}$  is by definition a simplicial object in  $\Delta^\circ \mathcal{C}$ . Then, the category of bisimplicial objects in  $\mathcal{C}$  is  $\Delta^\circ \Delta^\circ \mathcal{C} \simeq (\Delta \times \Delta)^\circ \mathcal{C}$ .

Given  $Z \in \Delta^\circ \Delta^\circ \mathcal{C}$  we will denote

$$\begin{aligned} d_i^{(1)} &= Z(\partial_i, Id) : Z_{n,m} \rightarrow Z_{n-1,m} & d_i^{(2)} &= Z(Id, \partial_i) : Z_{n,m} \rightarrow Z_{n,m-1} \\ s_j^{(1)} &= Z(\sigma_j, Id) : Z_{n,m} \rightarrow Z_{n+1,m} & s_j^{(2)} &= Z_{n,m} \rightarrow Z_{n,m+1} . \end{aligned}$$

Now we introduce some remarks that will be useful along these notes, as well as their dual constructions in the cosimplicial case.

**(1.1.19)** The diagonal functor  $D : \Delta^\circ \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}$  is the functor induced by composition with  $\Delta \rightarrow \Delta \times \Delta$ ,  $[n] \rightarrow ([n], [n])$ .

Then, given  $Z \in \Delta^\circ \Delta^\circ \mathcal{C}$ ,  $(DZ)_n = Z_{n,n}$  and  $(DZ)(\theta) = Z(\theta, \theta)$ ,  $\forall n \geq 0$  and  $\theta$  in  $\Delta$ .

**(1.1.20)** The index swapping in  $\Delta^\circ \Delta^\circ \mathcal{C}$  gives rise to a canonical functor  $\Gamma : \Delta^\circ \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \Delta^\circ \mathcal{C}$ , with

$$(\Gamma Z)_{n,m} = Z_{m,n} \quad (\Gamma Z)(\alpha, \beta) = Z(\beta, \alpha) ,$$

if  $n, m \geq 0$  and  $\alpha, \beta$  are morphisms in  $\Delta$ .

It holds that  $D \circ \Gamma = D$ .

**(1.1.21)** Each functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  induces by composition a functor between the respective categories of simplicial objects of  $\mathcal{C}$  and  $\mathcal{C}'$ , that will be denoted by  $\Delta^\circ F : \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}'$ . Then  $(\Delta^\circ F(X))_n = F(X_n) \ \forall n \geq 0$ .

**(1.1.22)** Let

$$- \times \Delta : \mathcal{C} \longrightarrow \Delta^\circ \mathcal{C}$$

be the simplicial constant functor. More concretely,  $X \times \Delta : \Delta^\circ \rightarrow \mathcal{C}$  is just the constant functor equal to  $X$

$$(X \times \Delta)_n = X \ \forall n \geq 0 \ ; \ (X \times \Delta)(f) = Id_X \ \forall \text{ morphism } f \text{ of } \Delta$$

Note that if  $*$  is a final object (resp. initial) in  $\mathcal{C}$ , so is  $* \times \Delta$  in  $\Delta^\circ \mathcal{C}$ .

**(1.1.23)** The functor  $- \times \Delta : \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}$  induces the functors

$$- \times \Delta, \ \Delta \times - : \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \Delta^\circ \mathcal{C} .$$

Given  $X$  in  $\Delta^\circ \mathcal{C}$  then

$$(X \times \Delta)_{n,m} = X_n \text{ and } (\Delta \times X)_{n,m} = X_m \ \forall n, m \geq 0 .$$

**REMARK 1.1.24.** The category  $\Delta^\circ \mathcal{C}$  inherits most of the properties satisfied by  $\mathcal{C}$ . For instance, if  $\mathcal{C}$  has finite coproducts the same holds for  $\Delta^\circ \mathcal{C}$  through:

$$(X \coprod Y)_n = X_n \coprod Y_n \ ; \ (X \coprod Y)(f) = X(f) \coprod Y(f) .$$

Analogously,  $\Delta^\circ \mathcal{C}$  is additive (resp. monoidal, abelian, complete, cocomplete, etc) if  $\mathcal{C}$  is so.

## 1.2 Augmented simplicial objects

**DEFINITION 1.2.1.** Let  $\Delta_+$  be the category whose objects are the ordinal numbers  $[n] = \{0, \dots, n\}$ ,  $n \geq -1$ , where  $[-1] = \emptyset$ , and whose morphisms are the (weak) monotone functions.

Then  $\Delta_+$  contains  $\Delta$  as a full subcategory.

Denote by  $\partial_0 : [-1] \rightarrow [0]$  the trivial morphism, that will be considered as a

face morphism. As in the case of  $\Delta$ , it also holds that every morphism has a unique factorization in terms of the face and degeneracy maps. These maps satisfy in the same way the simplicial identities.

**REMARK 1.2.2.** Analogously to 1.1.3,  $\Delta_+$  is a skeletal subcategory of  $\widehat{\Delta}_+$ , that is the category of (eventually empty) ordered sets and monotone functions. We will also denote by  $i : \Delta_+ \rightarrow \widehat{\Delta}_+$  and  $p : \widehat{\Delta}_+ \rightarrow \Delta_+$  the inclusion and its quasi-inverse.

The operation  $+$  :  $\widehat{\Delta}_+ \times \widehat{\Delta}_+ \rightarrow \widehat{\Delta}_+$  is defined as (1.1.4), and it makes  $\widehat{\Delta}_+$  (then also  $\Delta_+$ ) into a (strong) monoidal category (cf. [ML] p.171).

**DEFINITION 1.2.3.** An augmented simplicial object in a category  $\mathcal{C}$  is a functor  $X^+ : (\Delta_+)^{\circ} \rightarrow \mathcal{C}$ .

Denote by  $\Delta_+^{\circ}\mathcal{C}$  the category of augmented simplicial objects in  $\mathcal{C}$ .

Therefore, an augmented simplicial object  $X^+ \in \Delta_+^{\circ}\mathcal{C}$  is characterized by the data  $\{X_n, d_i, s_j\}$ , where  $X_n = X([n])$ ,  $d_i = X(\partial_i)$  and  $s_j = X(\sigma_j)$ . That is,  $X$  is represented by the following diagram

$$\begin{array}{ccccccc} X_{-1} & \xleftarrow{d_0} & X_0 & \xrightleftharpoons[d_0]{d_1} & X_1 & \xrightleftharpoons[\quad]{\quad} & X_2 & \xrightleftharpoons[\quad]{\quad} & X_3 & \cdots \end{array}$$

$\underbrace{\quad}_{s_0}$

**DEFINITION 1.2.4.** Given a simplicial object  $X \in \Delta^{\circ}\mathcal{C}$ , an augmentation of  $X$  is an augmented simplicial object  $X^+ \in \Delta_+^{\circ}\mathcal{C}$  such that its image under the forgetful functor  $U : \Delta_+^{\circ}\mathcal{C} \rightarrow \Delta^{\circ}\mathcal{C}$  induced by restriction is just  $X$ .

**REMARK 1.2.5.** Given  $X \in \Delta^{\circ}\mathcal{C}$ , an augmentation of  $X$  is a pair  $(X_{-1}, d_0)$  where  $X_{-1}$  is an object in  $\mathcal{C}$ , and  $d_0 : X_0 \rightarrow X_{-1}$  is such that  $d_0 d_0 = d_1 d_0 : X_1 \rightarrow X_{-1}$ .

If  $\mathcal{C}$  has a final object 1, then every simplicial object  $X$  has a trivial augmentation, taking  $X_{-1} = 1$ . Hence, we have the functor  $\Delta^{\circ}\mathcal{C} \rightarrow \Delta_+^{\circ}\mathcal{C}$ , right adjoint of  $U : \Delta_+^{\circ}\mathcal{C} \rightarrow \Delta^{\circ}\mathcal{C}$ .

**PROPOSITION 1.2.6.**

i) Given an object  $S$  in  $\mathcal{C}$  and a morphism  $\epsilon : X \rightarrow S \times \Delta$  in  $\Delta^{\circ}\mathcal{C}$ , then  $\epsilon_0 : X_0 \rightarrow S$  is an augmentation of  $X$ , and the correspondence  $\epsilon \rightarrow \epsilon_0$  is a bijection

$$\text{Hom}_{\Delta^{\circ}\mathcal{C}}(X, S \times \Delta) \simeq \{X^+ \in (\Delta_+)^{\circ}\mathcal{C} \text{ with } X_{-1} = S \text{ and } UX^+ = X\}$$

ii) The functor  $- \times \Delta : \mathcal{C} \rightarrow \Delta^{\circ}\mathcal{C}$  is left adjoint to  $\Delta^{\circ}\mathcal{C} \rightarrow \mathcal{C}$ ,  $X. \rightarrow X_0$ . That is,

$$\text{Hom}_{\Delta^{\circ}\mathcal{C}}(S \times \Delta, X) \simeq \text{Hom}_{\mathcal{C}}(S, X_0) .$$

*Proof.* *i)* Clearly, if  $\epsilon : X \rightarrow S \times \Delta$ , then  $d_0 = \epsilon_0 : X_0 \rightarrow S$  is an augmentation of  $X$ . Conversely, if  $d_0 : X_0 \rightarrow S$  is an augmentation of  $X$ , then  $\epsilon_n = (d_0)^{n+1} : X_n \rightarrow S$  defines a morphism  $\epsilon : X \rightarrow S \times \Delta$ . Moreover, it follows by induction that  $\epsilon_n = (d_0)^{n+1}$ .

To check *ii)*, it is enough to note that given  $\epsilon : S \times \Delta \rightarrow X$  then  $\epsilon_n = (s_0)^n \epsilon_0$ .  $\square$

Two relevant examples of augmented simplicial object are the “decalage” objects associated with any simplicial object (see [III]).

**(1.2.7)** Consider the functor  $F : \widehat{\Delta} \rightarrow \widehat{\Delta}$  (resp.  $G : \widehat{\Delta} \rightarrow \widehat{\Delta}$ ) given by  $F(E) = [0] + E$  (resp.  $G(E) = E + [0]$ ), that is, the ordered set obtained from  $E$  by adding a smallest (resp. greatest) element.

If we work in the category  $\Delta$ , the functor  $F : \Delta \rightarrow \Delta$  maps  $[n]$  into  $[n+1]$  and if  $\theta$  is a morphism  $\Delta$ ,  $F(\theta)(0) = 0$  and  $F(\theta)(i) = \theta(i-1) + 1$ , if  $i > 0$ .

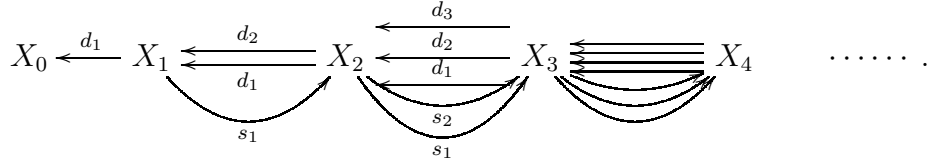
On the other hand, the functor  $G : \Delta \rightarrow \Delta$  is such that  $G([n]) = [n+1]$  and if  $\theta : [n] \rightarrow [m]$  then  $G(\theta)(i) = i$  if  $i < n+1$ ,  $G(n+1) = m+1$ .

**DEFINITION 1.2.8** (“Decalage” objects).

The “lower decalage functor”,  $dec_1 : \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}$ , is defined by composing with  $F$ . If  $X : \Delta^\circ \rightarrow \mathcal{C}$ , then  $X \circ F$  is obtained by “forgetting the first face and degeneracy maps” of  $X$

$$(dec_1(X))_n = X_{n+1} \quad (dec_1(X))(d_i) = d_{i+1} \quad (dec_1(X))(s_j) = s_{j+1} .$$

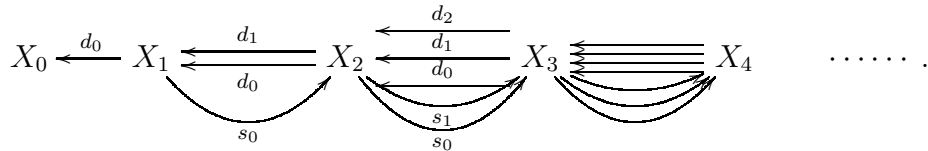
The morphism  $d_1 : X_1 \rightarrow X_0$  gives rise to the augmentation  $dec_1(X) \rightarrow X_0 \times \Delta$  given by



In the same way, by composition with  $G$  we obtain the functor “upper decalage”,  $dec^1(X) : \Delta^\circ \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}$  that consists of “forgetting the last face and degeneracy maps”

$$(dec^1(X))_n = X_{n+1} \quad (dec^1(X))(d_i) = d_i \quad (dec^1(X))(s_j) = s_j .$$

Therefore,  $d_0 : X_1 \rightarrow X_0$  produces the augmentation  $dec^1(X) \rightarrow X_0 \times \Delta$



### 1.2.1 Simplicial homotopy and extra degeneracy

We will remind the relationship between an augmentation with an extra degeneracy and the notion of homotopy between simplicial morphisms (cf. [B], p. 78).

First we give the combinatorial definition of homotopic morphisms in (cf. [May] §5).

**DEFINITION 1.2.9** (Simplicial homotopy).

The morphism  $f : X \rightarrow Y$  in  $\Delta^\circ \mathcal{C}$  is homotopic to  $g : X \rightarrow Y$ ,  $f \sim g$ , if there exists morphisms  $h_i : X_n \rightarrow Y_{n+1}$ ,  $i = 0, \dots, n$  satisfying the following identities

$$\begin{aligned} i) \quad & d_0 h_0 = f, \quad d_{n+1} h_n = g \\ ii) \quad & d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j \\ d_j h_{i-1} & \text{if } i = j \geq 1 \\ h_j d_{i-1} & \text{if } i > j + 1 \end{cases} \\ iii) \quad & s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j. \end{cases} \end{aligned}$$

Note that the relation  $\sim$  is not symmetric.

**DEFINITION 1.2.10.**

**1.** An augmentation  $X \rightarrow X_{-1} \times \Delta$  has a “lower” extra degeneracy if there exists morphisms  $s_{-1}^n = s_{-1} : X_n \rightarrow X_{n+1}$  in  $\mathcal{C}$  for all  $n \geq -1$  such that the following simplicial identities hold

$$d_0 s_{-1} = Id \quad d_{i+1} s_{-1} = s_{-1} d_i \quad s_j s_{-1} = s_{-1} s_{j-1} \quad \forall i \geq 0, \quad j \geq 0, \quad (1.4)$$

where  $\epsilon_0 = d_0 : X_0 \rightarrow X_{-1}$ .

**2.** Dually, an augmentation  $X \rightarrow X_{-1} \times \Delta$  has an “upper” extra degeneracy if there exists morphisms  $s_{n+1}^n = s_{n+1} : X_n \rightarrow X_{n+1}$  in  $\mathcal{C}$  for all  $n \geq -1$  such that

$$d_{n+1} s_{n+1} = Id \quad d_i s_{n+1} = s_n d_i \quad s_j s_{n+1} = s_{n+2} s_j \quad \forall i \leq n, \quad j \leq n + 1.$$

**EXAMPLE 1.2.11.** Following the notations introduced in 1.2.8, it is clear that the augmentation  $dec_1(X) \rightarrow X_0 \times \Delta$  has a “lower” extra degeneracy, consisting of the forgotten degeneracy  $s_0 : X_n \rightarrow X_{n+1}$ .

Analogously,  $dec^1(X) \rightarrow X_0 \times \Delta$  has an “upper” extra degeneracy  $s_n : X_n \rightarrow X_{n+1}$ .



**PROPOSITION 1.2.12** ([B] cap. 3, 3.2).

*An augmentation  $\epsilon : X \rightarrow X_{-1} \times \Delta$  has a lower extra degeneracy if and only if there exists  $\zeta : X_{-1} \times \Delta \rightarrow X$  such that  $Id_X \sim \zeta\epsilon$  and  $\epsilon\zeta = Id_{X_{-1} \times \Delta}$ .*

*Dually,  $\epsilon$  has an upper extra degeneracy if and only if there exists  $\zeta : X_{-1} \times \Delta \rightarrow X$  such that  $\zeta\epsilon \sim Id_X$  and  $\epsilon\zeta = Id_{X_{-1} \times \Delta}$ .*

### 1.3 Total object of a biaugmented bisimplicial object

In this section  $\mathcal{D}$  denotes a category with finite coproducts.

In this case, one can consider the classical cone object associated with a morphism  $f : X \rightarrow Y$  in  $\Delta^\circ \mathcal{D}$ , as well as the classical cylinder object associated with  $X$ . It turns out that both classical objects are particular cases of the “total” functor developed in this section.

In fact, the total functor can be seen as the simplicial analogue of the total chain complex associated with a double chain complex in an additive category. Even though this total functor is an extremely natural construction, the author could not find it in the literature.

Next we introduce this combinatorial construction associated with any biaugmented bisimplicial object  $Z$ , although we will only use some particular cases of  $Z$  in this work.

In the cosimplicial setting, all dual constructions and properties can be established.

**DEFINITION 1.3.1** (biaugmented bisimplicial object).

Let  $2 - \Delta_+$  be the full subcategory of  $\Delta_+ \times \Delta_+$  whose objects are the pairs  $([n], [m]) \in \Delta_+ \times \Delta_+$  such that  $[n]$  and  $[m]$  are not both empty.

A biaugmented bisimplicial object  $Z$  is a functor  $Z : 2 - \Delta_+^\circ \rightarrow \mathcal{D}$ , or equivalently a diagram  $Z_{-1,\cdot} \leftarrow Z_{\cdot,\cdot}^+ \rightarrow Z_{\cdot,-1}$ , where  $Z_{-1,\cdot}$ ,  $Z_{\cdot,\cdot}^+$  and  $Z_{\cdot,-1}$  are the respective restrictions of  $Z$  to  $[-1] \times \Delta$ ,  $\Delta \times \Delta$  and  $\Delta \times [-1]$ .

Hence  $Z$  is a diagram of  $\mathcal{D}$  of the form

(1.5)

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
Z_{-1,2} & \longleftarrow & Z_{0,2} & \rightleftharpoons & Z_{1,2} & \rightleftharpoons & Z_{2,2} & \rightleftharpoons & Z_{3,2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
Z_{-1,1} & \longleftarrow & Z_{0,1} & \rightleftharpoons & Z_{1,1} & \rightleftharpoons & Z_{2,1} & \rightleftharpoons & Z_{3,1} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
Z_{-1,0} & \longleftarrow & Z_{0,0} & \rightleftharpoons & Z_{1,0} & \rightleftharpoons & Z_{2,0} & \rightleftharpoons & Z_{3,0} & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & Z_{0,-1} & \rightleftharpoons & Z_{1,-1} & \rightleftharpoons & Z_{2,-1} & \rightleftharpoons & Z_{3,-1} & \cdots
\end{array}$$

We will denote by  $2 - \Delta_+^\circ \mathcal{D}$  the category of biaugmented bisimplicial objects, whose morphisms are natural transformations between functors.

**(1.3.2)** Again,  $2 - \Delta_+^\circ \mathcal{D}$  is canonically equivalent to the category  $2 - \widehat{\Delta}_+^\circ \mathcal{D}$ , where  $2 - \widehat{\Delta}_+$  is the full subcategory of  $\widehat{\Delta}_+ \times \widehat{\Delta}_+$  having as morphisms the pairs  $(E, F)$  of ordered sets  $E$  and  $F$  different from  $(\emptyset, \emptyset)$ .

The functors giving rise to this equivalence will be denoted by  $\mathcal{P} : 2 - \Delta_+^\circ \mathcal{D} \rightarrow 2 - \widehat{\Delta}_+^\circ \mathcal{D}$  and  $\mathcal{I} : 2 - \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow 2 - \Delta_+^\circ \mathcal{D}$ .

Next we will exhibit two examples of biaugmented bisimplicial object. First, we introduce the “total decalage” object associated with a simplicial object (cf. [III], p.7).

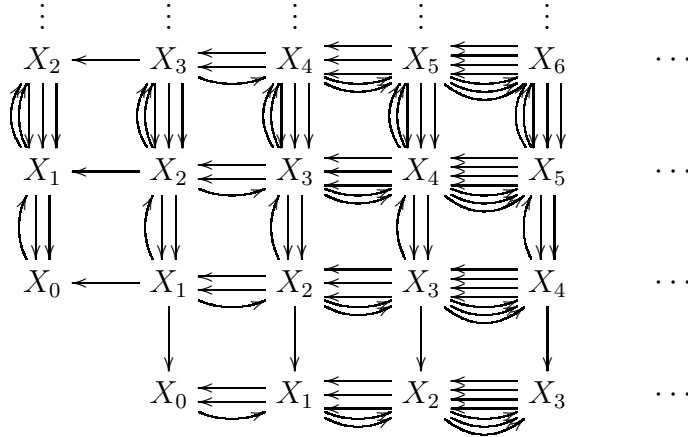
**(1.3.3)** The ordered sum of sets (see 1.2.2),  $+ : \widehat{\Delta}_+ \times \widehat{\Delta}_+ \rightarrow \widehat{\Delta}_+$ , can be restricted to

$$\wp : 2 - \Delta_+ \rightarrow \Delta ,$$

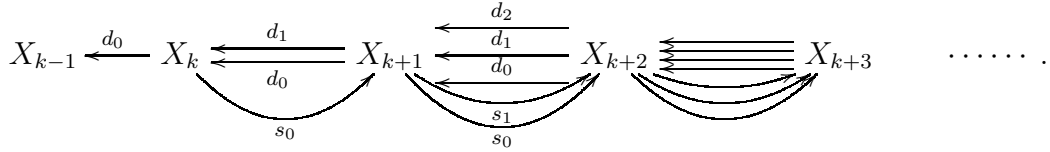
with  $\wp([n], [m]) = [n] + [m] = [n+m+1]$ , where  $[n]$  is identified with  $\{0, \dots, n\} \subset [n+m+1]$  and  $[m]$  with  $\{n+1, \dots, n+m+1\} \subset [n+m+1]$ .

**EXAMPLE 1.3.4.** The functor *total decalage*  $Dec : \Delta^\circ \mathcal{D} \rightarrow 2 - \Delta_+^\circ \mathcal{D}$  is defined by composition with  $\wp : 2 - \Delta_+ \rightarrow \Delta$ .

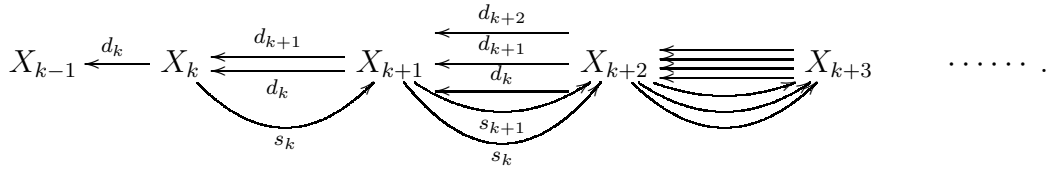
If  $X$  is a simplicial object, the biaugmented bisimplicial object  $Dec(X)$  consists of



Following the notations introduced in 1.2.8, the rows of the diagram are the augmented simplicial objects  $dec^k(X) \rightarrow X_{k-1}$ ,  $k \geq 1$ , where  $dec^k(X) = dec^1(\dots dec^1(X))$  is obtained from  $X$  by forgetting the last  $k$  face and degeneracy maps, that is



In the same way, the columns are the augmented simplicial objects  $dec_k(X) \rightarrow X_{k-1}$ , this time obtained by forgetting the first face and degeneracy maps of  $X$ , that is



Analogously,  $\widehat{Dec} : \widehat{\Delta}^\circ \mathcal{D} \rightarrow 2 - \widehat{\Delta}^\circ \mathcal{D}$  is defined as  $[\widehat{Dec}(X)](E, F) = X(E + F)$ , and it holds that  $Dec = \mathcal{I} \widehat{Dec} \mathcal{P}$  (see 1.3.2, 1.1.13).

**EXAMPLE 1.3.5.** Let  $f : X \rightarrow Y$  be a morphism in  $\Delta^\circ \mathcal{D}$  and  $\epsilon : X \rightarrow X_{-1} \times \Delta$  an augmentation of  $X$ .

Hence,  $X$  gives rise to the bisimplicial object  $Z^+ = \Delta \times X$ , that is,  $Z_{i,j}^+ = X_j$ . Moreover,  $f$  and  $\epsilon$  define the augmentations  $Z_{i,j}^+ \rightarrow Z_{-1,j} = Y_j$  and  $Z_{i,j}^+ \rightarrow Z_{i,-1} = X_{-1}$ .

Therefore the diagram  $Z = Z(f, \epsilon) : Z_{-1,\cdot} \leftarrow Z_{\cdot,\cdot}^+ \rightarrow Z_{\cdot,-1}$  is a biaugmented

bisimplicial object. Consequently, given  $([n], [m]), ([n'], [m'])$  in  $2 - \Delta_+$  and a morphism  $(\theta, \theta') : ([n'], [m']) \rightarrow ([n], [m])$ , we have that

$$Z_{n,m} = \begin{cases} Y_m & \text{if } n = -1 \\ X_m & \text{if } n \neq -1 \end{cases} \quad Z(\theta, \theta') = \begin{cases} Y(\theta') & \text{if } n = -1 \text{ (hence } n' = -1) \\ X(\theta') & \text{if } n' \neq -1 \text{ (hence } n \neq -1) \\ f_{m'} X(\theta') & \text{if } n' = -1 \text{ and } n \neq -1 \end{cases}$$

Visually,  $Z(f, \epsilon)$  is the following diagram

(1.6)

where the horizontal morphisms are either  $f_n : X_n \rightarrow Y_n$  or the identity  $X_n \rightarrow X_n$ , whereas the vertical morphisms are either the face and degeneracy maps of  $X$  or  $\epsilon_0 = d_0 : X_0 \rightarrow X_{-1}$ .

Clearly  $Z(f, \epsilon)$  is natural with respect to  $(f, \epsilon)$ . In addition, since  $\epsilon : X \rightarrow X_{-1} \times \Delta$  is characterized by  $\epsilon_0$  (see 1.2.6), then  $Z(f, \epsilon)$  preserves all the information contained in  $(f, \epsilon)$ .

**DEFINITION 1.3.6** (Total object of a biaugmented bisimplicial object).

The functor  $Tot : 2 - \Delta_+ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  is defined as follows. If  $Z$  is a biaugmented bisimplicial object, then  $Tot(Z)$  is in degree  $n$  the coproduct of the  $n$ -th antidiagonal of (1.5), that is

$$Tot(Z)_n = \coprod_{i+j=n-1} Z_{i,j}.$$

The face maps in  $Tot(Z)$  are defined as coproducts of the morphisms in (1.5) going from the  $n$ -th antidiagonal to the  $n - 1$ -th one, and analogously for the degeneracy maps.

More specifically, set  $\theta^{(1)} = Z(\theta, Id) : Z_{n,k} \rightarrow Z_{m,k}$  and  $\theta^{(2)} = Z(Id, \theta) : Z_{k,n} \rightarrow$

$Z_{k,m}$  for every  $\theta : [m] \rightarrow [n]$  and  $k \geq -1$ .

The face maps  $d_k : Tot(Z)_n \rightarrow Tot(Z)_{n-1}$  are given by

$$d_k|_{Z_{i,j}} = \begin{cases} d_k^{(1)} : Z_{i,j} \rightarrow Z_{i-1,j} & \text{if } i \geq k \\ d_{k-i-1}^{(2)} : Z_{i,j} \rightarrow Z_{i,j-1} & \text{if } i < k, \end{cases}.$$

The degeneracy maps  $s_k : Tot(Z)_n \rightarrow Tot(Z)_{n+1}$  are

$$s_k|_{Z_{i,j}} = \begin{cases} s_k^{(1)} : Z_{i,j} \rightarrow Z_{i+1,j} & \text{if } i \geq k \\ s_{k-i-1}^{(2)} : Z_{i,j} \rightarrow Z_{i,j+1} & \text{if } i < k, \end{cases}.$$

**(1.3.7)** Consider  $Z \in (2 - \Delta_+)^{\circ} \mathcal{D}$ . The canonical maps

$$Z_{-1,n} \rightarrow \coprod_{i+j=n-1} Z_{i,j} \text{ and } Z_{n,-1} \rightarrow \coprod_{i+j=n-1} Z_{i,j}$$

give rise to the following canonical morphisms in  $\Delta^{\circ} \mathcal{D}$

$$Z_{-1,\cdot} \rightarrow Tot(Z) \leftarrow Z_{\cdot,-1},$$

that are natural in  $Z$ .

**(1.3.8)** The functor  $Tot$  can be extended to  $Tot_+ : (\Delta_+ \times \Delta_+)^{\circ} \mathcal{D} \rightarrow (\Delta_+)^{\circ} \mathcal{D}$  as follows.

If  ${}_+Z \in (\Delta_+ \times \Delta_+)^{\circ} \mathcal{D}$  and  $Z$  if the restriction of  ${}_+Z$  to  $2 - \Delta_+$  then the morphism

$$d_0 = d_0^{(2)} \sqcup d_0^{(1)} : Z_{-1,0} \sqcup Z_{0,-1} \rightarrow {}_+Z_{-1,-1}$$

is an augmentation of  $Tot(Z)$ .

**REMARK 1.3.9.** As far as the author knows, the functors  $Tot$  and  $Tot_+$  do not appear in the literature.

However, a particular case of  $Tot_+$  is introduced in [EP], that is called the “join” of two augmented simplicial sets. If  $X \rightarrow X_{-1}$  and  $Y \rightarrow Y_{-1}$  are augmented simplicial sets then their join is just the image under the total functor of  ${}_+Z = \{X_n \times Y_m\}_{n,m \geq -1}$ . In loc. cit. the associativity of this join is stated. This associativity can be deduced as well from (the augmented version) of proposition 1.4.9.

Hence, the notion of join is completely similar to the one of tensor product of two chain complexes  $A$  and  $B$  of modules over a commutative ring  $R$ , since the tensor product of  $A$  and  $B$  is just the total chain complex associated with the double complex  $\{A_n \otimes_R B_m\}_{n,m}$ .

The definition given above for  $Tot$  is purely combinatorial, we introduced it in this way because we will need the formulae for computations. However, this construction can be understood in a more intuitive way if we consider the following equivalent definitions.

**(1.3.10)** The functor  $Tot : 2 - \Delta_+^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  can be described as follows.

- Consider the category  $\Delta/[1]$  with objects the pairs  $([n], \sigma)$ , where  $\sigma : [n] \rightarrow [1]$  is a morphism in  $\Delta$ . A morphism  $\theta : ([n], \sigma) \rightarrow ([m], \rho)$  is  $\theta : [n] \rightarrow [m]$  in  $\Delta$  such that  $\rho\theta = \sigma$ . We will denote  $([n], \sigma)$  by  $\sigma$  if  $[n]$  is understood.
- If  $\sigma : [n] \rightarrow [1]$ , let  $i_\sigma \in \{0, \dots, n+1\}$  be such that  $\{i_\sigma, \dots, n\} = \sigma^{-1}(1)$  if it is non empty, and  $i_\sigma = n+1$  if  $\sigma^{-1}(1) = \emptyset$ . Note that the correspondence  $\sigma \rightarrow i_\sigma$  is a bijection. Moreover, we will identify  $[i_\sigma - 1]$  with  $\sigma^{-1}(0)$ , as well as  $[n - i_\sigma]$  with  $\sigma^{-1}(1)$  (after relabelling in a suitable way).
- Let  $\Psi : \Delta/[1] \rightarrow 2 - \Delta_+$  be the functor defined by  $\Psi(\sigma) = [i_\sigma - 1] \times [n - i_\sigma]$ . Given  $\theta : ([n], \sigma) \rightarrow ([m], \rho)$ , since  $\theta(\sigma^{-1}(j)) \subseteq \rho^{-1}(j)$  for  $j = 0, 1$  then  $\theta|_{\sigma^{-1}(0)} : [i_\sigma - 1] \rightarrow [i_\rho - 1]$ , and  $\theta|_{\sigma^{-1}(1)} : [n - i_\sigma] \rightarrow [m - i_\rho]$  are morphisms in  $\Delta_+$ .  
Hence, define  $\Psi(\theta) = \theta|_{\sigma^{-1}(0)} \times \theta|_{\sigma^{-1}(1)} : [i_\sigma - 1] \times [n - i_\sigma] \rightarrow [i_\rho - 1] \times [m - i_\rho]$ .
- If  $Z$  is a biaugmented bisimplicial object,  $Tot(Z)$  is the simplicial object defined in degree  $n$  as

$$Tot(Z)_n = \coprod_{\sigma : [n] \rightarrow [1]} Z_{i_\sigma - 1, n - i_\sigma}$$

If  $\rho : [n] \rightarrow [1]$  and  $\theta : [n] \rightarrow [m]$ , then  $\theta : ([n], \rho\theta) \rightarrow ([m], \rho)$  is in  $\Delta/[1]$ , and the restriction of  $Tot(Z)(\theta)$  to  $Z_{i_\rho - 1, m - i_\rho}$  is

$$Z(\Psi(\theta)) : Z_{i_\rho - 1, m - i_\rho} \rightarrow Z_{i_{\rho\theta} - 1, n - i_{\rho\theta}}$$

**(1.3.11)** In terms of  $\widehat{\Delta}_+$ ,  $\widehat{Tot} : 2 - \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow \widehat{\Delta}^\circ \mathcal{D}$  is defined as

$$[\widehat{Tot}(Z)](E) = \coprod_{E = E_0 + E_1} Z(E_0, E_1) ,$$

where the coproduct is indexed over the set of pairs  $(E_0, E_1) \in 2 - \widehat{\Delta}_+$  such that  $E$  is the ordered sum of  $E_0$  and  $E_1$  (see 1.2.2).

Let  $f : E' \rightarrow E$  be a morphism in  $\widehat{\Delta}$  and  $(E_0, E_1)$  an object in  $2 - \widehat{\Delta}_+$  such that  $E_0 + E_1 = E$ .

Denote by  $E'_i$  the set  $f^{-1}(E_i)$  with the order induced by the one of  $E'$ , for  $i = 0, 1$ . It is clear that  $E' = E'_0 + E'_1$ , as well as  $f = f_0 + f_1$ , where  $f_i = f|_{E'_i} : E'_i \rightarrow E_i$  for  $i = 0, 1$ .

Then  $[\widehat{Tot}(Z)](f) : [\widehat{Tot}(Z)](E) \rightarrow [\widehat{Tot}(Z)](E')$  is defined as

$$[\widehat{Tot}(Z)](f)|_{Z(E_0, E_1)} = Z(f_0, f_1) : Z(E_0, E_1) \rightarrow Z(E'_0, E'_1) .$$

**(1.3.12)** The functor  $\widehat{Tot}$  can be also extended in a natural way to

$$\widehat{Tot}^+ : (\widehat{\Delta}_+ \times \widehat{\Delta}_+)^{\circ} \mathcal{D} \rightarrow \widehat{\Delta}_+^{\circ} \mathcal{D} ,$$

with  $[\widehat{Tot}^+(Z)](\emptyset) = Z(\emptyset, \emptyset)$ .

**PROPOSITION 1.3.13.** *The two definitions given for  $Tot$  coincide, and do define a functor  $Tot : 2 - \Delta_+^{\circ} \mathcal{D} \rightarrow \Delta^{\circ} \mathcal{D}$ .*

*In addition, these constructions correspond to  $\widehat{Tot} : 2 - \widehat{\Delta}_+^{\circ} \mathcal{D} \rightarrow \widehat{\Delta}^{\circ} \mathcal{D}$  under the canonical equivalence of the categories  $\Delta_+$  and  $\widehat{\Delta}_+$ . That is, under the notations of 1.3.2 and 1.1.13 we have that*

$$Tot = \mathcal{I} \circ \widehat{Tot} \circ \mathcal{P} .$$

*Proof.* Let us see first the second statement.

Define  $\widehat{\Delta}/[1]$  as 1.3.10, as well as  $\widehat{\Psi} : \widehat{\Delta}/[1] \rightarrow 2 - \widehat{\Delta}_+$ . More specifically, if  $\sigma : E \rightarrow [1]$ , then  $\widehat{\Psi}(\sigma) = \sigma^{-1}(0) \times \sigma^{-1}(1)$ .

Any decomposition  $E = E_0 + E_1$  in  $2 - \widehat{\Delta}_+$  determines in a unique way an object  $\sigma : E \rightarrow [1]$  in  $\widehat{\Delta}/[1]$  (just take  $\sigma(E_i) = i$ ,  $i = 0, 1$ ).

Hence 1.3.11 can be rewritten as

$$[\widehat{Tot}(Z)](E) = \coprod_{\sigma : E \rightarrow [1]} Z(\sigma^{-1}(0), \sigma^{-1}(1)) .$$

Moreover, if  $f : E' \rightarrow E$ , then the restriction of  $[Tot(Z)](f)$  to  $Z(\sigma^{-1}(0), \sigma^{-1}(1))$  coincides with  $Z(f|_{(f\sigma)^{-1}(0)}, f|_{(f\sigma)^{-1}(1)})$ . Consequently

$$[\mathcal{I} \circ \widehat{Tot} \circ \mathcal{P}(Z)]([n]) = \coprod_{\sigma : [n] \rightarrow [1]} (\mathcal{P}Z)(\sigma^{-1}(0), \sigma^{-1}(1)) = \coprod_{\sigma : [n] \rightarrow [1]} Z(p(\sigma^{-1}(0)), p(\sigma^{-1}(1))) ,$$

that is equal to  $Tot(Z)_n$ , and clearly the action of  $\mathcal{I} \circ \widehat{Tot} \circ \mathcal{P}(Z)$  and of  $Tot(Z)$  over the morphisms of  $\Delta$  coincides.

Secondly, let us check that the two definitions given for the total functor coincide. Note that any monotone function  $\sigma : [n] \rightarrow [1]$  is characterized by the integer  $i_\sigma \in \{0, n+1\}$  such that  $\sigma^{-1}(1) = \{i_\sigma, \dots, n\}$ .

It follows that the correspondence  $\sigma \rightarrow i_\sigma - 1$  is one-to-one between the sets

$$\{(i, j) \mid i + j = n - 1, i, j \geq -1\} \text{ and } \{(i_\sigma - 1, n - i_\sigma) \mid \sigma : [n] \rightarrow [1]\}.$$

Let  $\theta = \partial_k : [n-1] \rightarrow [n]$  and  $\rho : [n] \rightarrow [1]$  be morphisms in  $\Delta$ . We will compute

$$\Psi(\partial_k) : [i_{\rho\partial_k} - 1] \times [n - 1 - i_{\rho\partial_k}] \rightarrow [i_\rho - 1] \times [n - i_\rho].$$

If  $i_\rho \leq k$ , then  $i_{\rho\partial_k} = i_\rho$  and it holds that

$$\partial_k|_{(\rho\partial_k)^{-1}(0)} = Id : [i_\rho - 1] \rightarrow [i_\rho - 1] ; \partial_k|_{(\rho\partial_k)^{-1}(1)} = \partial_{k-i_\rho} : [n - i_\rho - 1] \rightarrow [n - i_\rho].$$

In this case  $[Tot(Z)](\partial_k)|_{Z_{i_\rho-1, n-i_\rho}} = d_{k-i_\rho}^{(2)} : Z_{i_\rho-1, n-i_\rho} \rightarrow Z_{i_\rho-1, n-i_\rho-1}$ .

Then, taking  $i = i_\rho - 1$ ,  $[Tot(Z)](\partial_k)|_{Z_{i_\rho-1, n-i_\rho}}$  corresponds to  $d_{k-i-1}^{(2)} : Z_{i,j} \rightarrow Z_{i,j-1}$ .

On the other hand, if  $i_\rho > k$ , then  $i_{\rho\partial_k} = i_\rho - 1$  and it can be checked analogously that  $[Tot(Z)](\partial_k)|_{Z_{i_\rho-1, n-i_\rho}} = Z(\partial_k, Id) : Z_{i_\rho-1, n-i_\rho} \rightarrow Z_{i_\rho-2, n-i_\rho}$ .

Hence, setting  $i = i_\rho - 1$  we have that  $[Tot(Z)](\partial_k)|_{Z_{i_\rho-1, n-i_\rho}}$  is  $d_k^{(1)} : Z_{i,j} \rightarrow Z_{i-1,j}$ .

The analogous equality involving the degeneracy maps  $\sigma_k : [n+1] \rightarrow [n]$  can be checked in a similar way.

It remains to see that  $Tot(Z) \in \Delta^\circ \mathcal{D}$ , that is, that  $Tot(Z)(\theta\theta') = Tot(Z)(\theta')Tot(Z)(\theta)$  for every composable morphisms  $\theta$  and  $\theta'$  in  $\Delta$ . The equality  $Tot(Z)(Id) = Id$  follows from the naturality of  $\Psi$ .

Finally, given  $\psi : Z \rightarrow S$  then  $Tot(\psi) : Tot(Z) \rightarrow Tot(S)$  is defined in degree  $n$  by  $Tot(\psi)|_{Z_{i,j}} = \psi_{i,j} : Z_{i,j} \rightarrow S_{i,j}$ . Clearly, it is a morphism between simplicial objects, natural in  $\psi$ .  $\square$

**REMARK 1.3.14.** Let  $\mathcal{I} : \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow \Delta_+^\circ \mathcal{D}$  and  $\mathcal{P} : (\Delta_+ \times \Delta_+)^\circ \mathcal{D} \rightarrow (\widehat{\Delta}_+ \times \widehat{\Delta}_+)^\circ \mathcal{D}$  be the equivalences of categories induced by  $i : \Delta_+ \rightarrow \widehat{\Delta}_+$  and  $p : \widehat{\Delta}_+ \rightarrow \Delta_+$ . Hence, it holds that  $Tot^+ = \mathcal{I} \circ \widehat{Tot}^+ \circ \mathcal{P}$ .

**PROPOSITION 1.3.15.** *The functor  $\widehat{Tot} : 2 - \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow \widehat{\Delta}^\circ \mathcal{D}$  is left adjoint to the total decalage functor  $\widehat{Dec} : \Delta^\circ \mathcal{D} \rightarrow 2 - \Delta_+^\circ \mathcal{D}$  introduced in 1.3.4.*

*Hence  $(Tot, Dec)$  is also an adjoint pair of functors.*

*Consequently, the functors  $\widehat{Tot}$  and  $Tot$  commutes with colimits and in particular with coproducts.*



*Proof.* Since  $\mathcal{P}$  and  $\mathcal{I}$  are quasi-inverse equivalences of categories, and since  $\widehat{Tot} = \mathcal{I} \circ \widehat{Tot} \circ \mathcal{P}$  and  $\widehat{Dec} = \mathcal{I} \circ \widehat{Dec} \circ \mathcal{P}$ , it follows from the adjunction  $(\widehat{Tot}, \widehat{Dec})$  that  $(Tot, Dec)$  is also an adjoint pair.

Consider  $Z \in 2 - \widehat{\Delta}_+^\circ \mathcal{D}$  and  $Y \in \widehat{\Delta}^\circ \mathcal{D}$ . Let us check that there is a canonical bijection

$$\mathrm{Hom}_{\widehat{\Delta}^\circ \mathcal{D}} \left( \widehat{Tot}(Z), Y \right) \simeq \mathrm{Hom}_{2 - \widehat{\Delta}_+^\circ \mathcal{D}} \left( Z, \widehat{Dec}(Y) \right) .$$

A morphism  $F : \widehat{Tot}(Z) \rightarrow Y$  consists of a collection of morphisms in  $\mathcal{D}$

$$G(E_0, E_1) = F(E)|_{Z(E_0, E_1)} : Z(E_0, E_1) \rightarrow Y(E) = [\widehat{Dec}(Y)](E_0, E_1)$$

for every equality of ordered sets  $E = E_0 + E_1$ .

Moreover,  $G$  is natural in  $(E_0, E_1)$  since  $F$  is natural in  $E$ . To see this, given an expression  $E = E_0 + E_1$ , it is enough to note that there exists a bijection between the set of morphisms  $(f_0, f_1) : (E'_0, E'_1) \rightarrow (E_0, E_1)$  in  $2 - \widehat{\Delta}_+$  and the morphisms  $f : E' \rightarrow E$ . To see this, set  $E' = E'_0 + E'_1$ ;  $f = f_0 + f_1 : E' \rightarrow E$  on one hand, and  $E'_i = f^{-1}(E_i)$ ;  $f_i = f|_{E'_i}$ ,  $i = 0, 1$  on the other hand.

The naturality of  $F$  means that for every ordered set  $E$  the following equality holds

$$F(E') \circ \widehat{Tot}(Z)(f) = Y(f) \circ F(E) : [\widehat{Tot}(Z)](E) \rightarrow Y(E') ,$$

and this happens if and only if this equality holds on each component  $Z(E_0, E_1)$  of  $\widehat{Tot}(Z)$ . That is to say, if and only if

$$\begin{aligned} G(E'_0, E'_1) \circ Z(f_0, f_1) &= (F(E') \circ \widehat{Tot}(Z)(f))|_{Z(E_0, E_1)} = \\ &= (Y(f) \circ F(E))|_{Z(E_0, E_1)} = [\widehat{Dec}(Y)](f_0, f_1) \circ G(E_0, E_1) \end{aligned}$$

and this happens if and only if  $G$  is natural with respect to each morphism  $(f_0, f_1)$ .  $\square$

**REMARK 1.3.16.** In [CR] the authors consider a “decalage” functor  $dec : \Delta^\circ Set \rightarrow \Delta^\circ \Delta^\circ Set$ , by forgetting the biaugmentation of the biaugmented bisimplicial object  $Dec(X)$  associated with  $X \in \Delta^\circ Set$ . In loc. cit. another “combinatorial” functor  $\Delta^\circ \Delta^\circ Set \rightarrow Set$  is also introduced. This functor is right adjoint to  $dec$  and is defined using fiber products in  $Set$  instead of coproducts.

Next we will see that the classical simplicial cone and cylinder objects are particular cases of the functor  $Tot$ .

**EXAMPLE 1.3.17** (Simplicial cone).

Assume that  $\mathcal{D}$  has a final object  $1$  and  $f : X \rightarrow Y$  is a morphism in  $\Delta^\circ \mathcal{D}$ . The

classical definition of cone object associated with  $f$  [DeIII] is  $Cf \in \Delta^\circ \mathcal{D}$ , with

$$(Cf)_n = Y_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_0 \sqcup 1 .$$

Let  $Z$  be the biaugmented bisimplicial object associated with  $f : X \rightarrow Y$  and to the trivial augmentation  $X \rightarrow 1 \times \Delta$  (see 1.3.5).

Hence  $Cf$  is just the total simplicial object of  $Z$ .

**EXAMPLE 1.3.18** ('Cubical' cylinder).

Given a simplicial object  $X$  in  $\mathcal{D}$ , let us remind the classical notion of cylinder associated with  $X$ , that will be denoted by  $\widetilde{Cyl}(X)$ . We will say that  $\widetilde{Cyl}(X)$  is the "cubical" cylinder object of  $X$ . It is defined in degree  $n$  as

$$\widetilde{Cyl}(X)_n = \coprod_{\sigma: [n] \rightarrow [1]} X_n^\sigma ,$$

where  $X_n^\sigma = X_n \ \forall \sigma$ . Given  $\theta : [m] \rightarrow [n]$  in  $\Delta$ ,  $\widetilde{Cyl}(\theta) : \widetilde{Cyl}(X)_n \rightarrow \widetilde{Cyl}(X)_m$  is

$$\widetilde{Cyl}(\theta)|_{X_n^\sigma} = X(\theta) : X_n^\sigma \rightarrow X_m^{\sigma\theta} .$$

It holds that  $\widetilde{Cyl}(X)$  is the total object of  $Dec(X) \in (2 - \Delta_+)^{\circ} \mathcal{D}$  (given in 1.3.4).

**REMARK 1.3.19.** We recall that the cubical cylinder object characterizes simplicial homotopies [[May] prop. 6.2].

In other words, let  $u_0, u_1 : [n] \rightarrow [1]$  be the morphisms such that  $u_i([n]) = i$ ,  $i = 0, 1$ . If  $X$  is a simplicial object, define  $I, J : X \rightarrow \widetilde{Cyl}(X)$  as

$$I_n = Id : X_n \rightarrow X_n^{u_1} \text{ and } J_n = Id : X_n \rightarrow X_n^{u_0} .$$

Given  $f, g : X \rightarrow Y$  in  $\Delta^\circ \mathcal{D}$ , we have that  $f \sim g$  (that is,  $f$  is homotopic to  $g$ ) if and only if there exists  $H : \widetilde{Cyl}(X) \rightarrow Y$  such that  $H \circ I = f$  and  $H \circ J = g$ .

## 1.4 Total object of $n$ -augmented $n$ -simplicial objects

In this section we will generalize in a natural way the functors  $Tot$  and  $\widehat{Tot}$  to the categories  $n - \Delta_+^\circ \mathcal{D}$  and  $n - \widehat{\Delta}_+^\circ \mathcal{D}$ , respectively. In addition, the functor  $Tot_n$  (resp.  $\widehat{Tot}_n$ ) can be obtain as well as iterations of  $Tot$  (resp.  $\widehat{Tot}$ ).

**DEFINITION 1.4.1.** Let  $n - \Delta_+$  be the full subcategory of  $\Delta_+ \times \cdots \times \Delta_+$  whose objects are  $([m_0], \dots, [m_{n-1}])$  with  $\sum m_k \neq -n$ . Analogously, let  $n - \widehat{\Delta}_+$  be the category defined using  $\widehat{\Delta}_+$  instead of  $\Delta_+$ .

Again, we will refer to the elements of the category  $n - \Delta_+^\circ \mathcal{D}$  of contravariant functors from  $n - \Delta_+$  to  $\mathcal{D}$  as  $n$ -augmented  $n$ -simplicial objects.

(1.4.2) Similarly to the case  $n = 2$ , an object  $T$  of  $3 - \Delta_+^\circ \mathcal{D}$  can be visualized as

$$\begin{array}{ccccc}
 & & T_{i,-1,k} & \longrightarrow & T_{i,-1,-1} \\
 & \swarrow & \uparrow & & \uparrow \\
 T_{-1,-1,k} & & & & \\
 \uparrow & & T_{i,j,k} & \longrightarrow & T_{i,j,-1} \\
 & \swarrow & & \searrow & \\
 T_{-1,j,k} & \longrightarrow & T_{-1,j,-1} & & 
 \end{array}$$

where the indexes  $i, j, k$  are positive integers.

**DEFINITION 1.4.3.** Define  $Tot_n : n - \Delta_+^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  as follows.

Let  $\Delta/[n-1]$  be the category defined as  $\Delta/[1]$  in 1.3.10.

Given  $\sigma : [m] \rightarrow [n-1]$  and  $k \in [n-1]$ , note that each ordered subset  $\sigma^{-1}(k)$  of  $[m]$  is isomorphic to a unique object  $[m_k(\sigma)]$  of  $\Delta_+$ .

If  $Z \in n - \Delta_+^\circ \mathcal{D}$ , set

$$[Tot_n(Z)]_m = \coprod_{\sigma : [m] \rightarrow [n-1]} Z_{m_0(\sigma), \dots, m_{n-1}(\sigma)} .$$

Moreover, if  $\theta : [m'] \rightarrow [m]$ , then  $\theta|_{(\sigma\theta)^{-1}(k)} : (\sigma\theta)^{-1}(k) \rightarrow (\sigma)^{-1}(k)$  induces a monotone function  $\theta_k : [m'_k(\sigma\theta)] \rightarrow [m_k(\sigma)]$ , and the restriction of  $[Tot_n(Z)](\theta)$  to  $Z_{m_0(\sigma), \dots, m_{n-1}(\sigma)}$  is

$$Z(\theta_0, \dots, \theta_{m-1}) : Z_{m_0(\sigma), \dots, m_{n-1}(\sigma)} \rightarrow Z_{m'_0(\sigma\theta), \dots, m'_{n-1}(\sigma\theta)} .$$

Next we provide a description of the total functor in terms of  $\widehat{\Delta}_+$ .

(1.4.4) The functor  $\widehat{Tot}_n : n - \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow \widehat{\Delta}^\circ \mathcal{D}$  is given by

$$[\widehat{Tot}_n(Z)](E) = \coprod_{E=E_0+\dots+E_{n-1}} Z(E_0, \dots, E_{n-1}) .$$

If  $f : E' \rightarrow E$  is a morphism in  $\widehat{\Delta}$  and  $E = E_0 + \dots + E_{n-1}$  then

$$E' = E'_0 + \dots + E'_{n-1} \quad \text{and} \quad f = f_0 + \dots + f_{n-1} ,$$

where  $E'_i = \theta^{-1}(E_i)$  and  $f_i = f|_{E'_i} : E'_i \rightarrow E_i$ , for  $i = 0, \dots, n-1$ .

Hence  $[\widehat{Tot}_n(Z)](f) : [\widehat{Tot}_n(Z)](E) \rightarrow [\widehat{Tot}_n(Z)](E')$  is

$$[\widehat{Tot}_n(Z)](f)|_{Z(E_0, \dots, E_{n-1})} = Z(f_0, \dots, f_{n-1}) : Z(E_0, \dots, E_{n-1}) \rightarrow Z(E'_0, \dots, E'_{n-1}) .$$

**REMARK 1.4.5.** As in 1.3.2, the equivalences of categories  $i : \Delta \rightarrow \widehat{\Delta}$ ,  $p : \widehat{\Delta} \rightarrow \Delta$  induce the quasi-inverse equivalences

$$\mathcal{I} : n - \widehat{\Delta}^\circ \mathcal{D} \rightarrow n - \Delta^\circ \mathcal{D} \quad \mathcal{P} : n - \Delta^\circ \mathcal{D} \rightarrow n - \widehat{\Delta}^\circ \mathcal{D} .$$

Then, the following analogue of 1.3.15 also holds

$$Tot_n = \mathcal{I} \circ \widehat{Tot}_n \circ \mathcal{P} .$$

**REMARK 1.4.6.** Since  $+: 2 - \widehat{\Delta}_+ \rightarrow \widehat{\Delta}$  is associative, we can consider the sum  $E_0 + \cdots + E_{n-1}$  with no need of fixing the order in which the sums are made. Consequently, the following property holds.

Consider a non empty ordered set  $E$ . It is clear that there is a bijection between the decompositions of  $E$  as an ordered sum of  $n$  of its subsets,  $E = E_0 + \cdots + E_{n-1}$ , and the decompositions  $E = E'_0 + E'_1$  together with two more decompositions  $E'_0 = E_0 + \cdots + E_{k-1}$  and  $E'_1 = E_k + \cdots + E_{n-1}$ , for a fixed  $0 \leq k \leq n-1$ .

Iterating this procedure, it follows that  $\widehat{Tot}_n$  (resp.  $Tot_n$ ) agrees with consecutive iterations of  $\widehat{Tot} = \widehat{Tot}_2$  (resp.  $Tot = Tot_2$ ). Let us see the case  $n = 3$  in more detail.

**(1.4.7)** Consider  $T \in 3 - \widehat{\Delta}_+^\circ \mathcal{D}$  and an ordered set  $E$ .

Roughly speaking,  $\coprod_{F=G+H} T(E, G, H)$  is the total object of  $T(E, -, -)$  evaluated at  $F$ .

However, when  $E \neq \emptyset$  it turns out that  $T(E, -, -) \in (\widehat{\Delta}_+ \times \widehat{\Delta}_+)^\circ \mathcal{D}$ , whereas  $T(\emptyset, -, -) \in 2 - \widehat{\Delta}_+^\circ \mathcal{D}$ . Therefore, we introduce the following notations.

Given  $E \in \widehat{\Delta}_+$ , we define

$$\widehat{Tot}^*(T(E, -, -)) = \begin{cases} \widehat{Tot}^+(T(E, -, -)) & \text{if } E \neq \emptyset \\ \widehat{Tot}(T(\emptyset, -, -)) & \text{if } E = \emptyset , \end{cases}$$

where  $\widehat{Tot}^+ : (\widehat{\Delta}_+ \times \widehat{\Delta}_+)^\circ \mathcal{D} \rightarrow \widehat{\Delta}_+^\circ \mathcal{D}$  is defined as  $Tot^+$  in 1.3.12.

We define also

$$\widehat{Tot}^*(T(-, -, E)) = \begin{cases} \widehat{Tot}^+(T(-, -, E)) & \text{if } E \neq \emptyset \\ \widehat{Tot}(T(-, -, \emptyset)) & \text{if } E = \emptyset . \end{cases}$$

**LEMMA 1.4.8.** *There exists functors  $\widehat{Tot}_{(0)}, \widehat{Tot}_{(2)} : 3 - \widehat{\Delta}_+^\circ \mathcal{D} \rightarrow 2 - \widehat{\Delta}_+^\circ \mathcal{D}$ , such that*

$$\begin{aligned} [\widehat{Tot}_{(0)}(T)](E, F) &= [\widehat{Tot}^*(T(E, -, -))](F) \quad \text{and} \\ [\widehat{Tot}_{(2)}(T)](E, F) &= [\widehat{Tot}^*(T(-, -, F))](E) \end{aligned}$$

for any  $T$  in  $3 - \widehat{\Delta}_+^\circ \mathcal{D}$ .

*Proof.* Firstly, it suffices to prove the statement for  $Tot_{(0)}(T)$ . In this case, if  $R \in 3 - \widehat{\Delta}_+^\circ \mathcal{D}$  is given by  $R(E, F, G) = T(G, F, E)$ , we have that  $Z = \widehat{Tot}_{(0)}(R)$  is a biaugmented bisimplicial object. Note that  $\widehat{Tot}_{(2)}(T)$  is just the object in  $2 - \widehat{\Delta}_+^\circ \mathcal{D}$  obtained by interchanging the indexes  $E$  and  $F$  in  $Z$ .

Let us check that  $\widehat{Tot}_{(0)}(T)$  is in fact in  $2 - \widehat{\Delta}_+^\circ \mathcal{D}$ . Given  $(f, g) : (E', F') \rightarrow (E, F)$ , then

$$[\widehat{Tot}_{(0)}(T)](f, g) : [\widehat{Tot}_{(0)}(T)](E, F) \rightarrow [\widehat{Tot}_{(0)}(T)](E', F')$$

is induced by  $T$  in a natural way.

To see this, we have that

$$[\widehat{Tot}_{(0)}(T)](E, F) = \coprod_{F=F_0+F_1} T(E, F_0, F_1) ; [\widehat{Tot}_{(0)}(T)](E', F') = \coprod_{F'=F'_0+F'_1} T(E', F'_0, F'_1) .$$

Given  $F_0$  and  $F_1$  with  $F = F_0 + F_1$ , set  $F'_i = g^{-1}(F_i)$ ,  $g_i = g|_{F'_i} : F'_i \rightarrow F_i$  for  $i = 0, 1$ .

Then

$$[\widehat{Tot}_{(0)}(T)](f, g)|_{T(E, F_0, F_1)} = T(f, g_0, g_1) : T(E, F_0, F_1) \rightarrow T(E', F'_0, F'_1) ,$$

and clearly  $\widehat{Tot}_{(0)}(T)$  is functorial in  $(f, g)$ .  $\square$

**PROPOSITION 1.4.9.** *The functors  $\widehat{Tot}_3$ ,  $\widehat{Tot} \circ \widehat{Tot}_{(0)}$  and  $\widehat{Tot} \circ \widehat{Tot}_{(2)}$  are isomorphic.*

*In other words, given  $T \in 3 - \widehat{\Delta}^\circ \mathcal{D}$ , there are canonical and functorial isomorphisms*

$$\widehat{Tot}_3(T) \simeq \widehat{Tot}(\widehat{Tot}_{(0)}^*(T)) \quad \widehat{Tot}_3(T) \simeq \widehat{Tot}(\widehat{Tot}_{(2)}^*(T)) .$$

*Proof.* By definition

$$[\widehat{Tot}_3(T)](E) = \coprod_{E=E_0+E_1+E_2} T(E_0, E_1, E_2) .$$

Clearly, the sets

$$\begin{aligned} &\{(E_0, E_1, E_2) \in 3 - \widehat{\Delta} \mid E = E_0 + E_1 + E_2\} \\ &\{(F, G) \times (G_0, G_1) \in (2 - \widehat{\Delta}) \times (2 - \widehat{\Delta}) \mid E = F + G \text{ and } G = G_0 + G_1\} \\ &\{(E, F) \times (E_0, E_1) \in (2 - \widehat{\Delta}) \times (2 - \widehat{\Delta}) \mid E = F + G \text{ and } F = F_0 + F_1\} \end{aligned}$$

are bijective. Hence, after reordering the terms in the coproduct defining  $[\widehat{Tot}_3(Z)](E)$  we obtain canonical isomorphisms

$$\begin{aligned} [\widehat{Tot}_3(T)](E) &\stackrel{\sigma_E}{\cong} \coprod_{E=F+G} \left( \coprod_{G=G_0+G_1} T(F, G_0, G_1) \right) = \coprod_{E=F+G} \widehat{Tot}_{(0)}^*(T)(F, G) = [\widehat{Tot}(\widehat{Tot}_{(0)}^*(T))](E) \\ [\widehat{Tot}_3(T)](E) &\stackrel{\rho_E}{\cong} \coprod_{E=F+G} \left( \coprod_{F=F_0+F_1} T(F_0, F_1, G) \right) = \coprod_{E=F+G} \widehat{Tot}_{(2)}^*(T)(F, G) = [\widehat{Tot}(\widehat{Tot}_{(2)}^*(T))](E) \end{aligned}$$

Therefore, for every  $f : E' \rightarrow E$  it holds that  $[\widehat{Tot}(\widehat{Tot}_{(0)}^*(T))](f) \circ \sigma_E = \sigma_{E'} \circ [\widehat{Tot}_3(T)](f)$  (and similarly for  $\rho$ ).

Consider  $E_i$ ,  $i = 0, 1, 2$  with  $E = E_0 + E_1 + E_2$ .

Then  $[\widehat{Tot}_3(T)](f)|_{T(E_0, E_1, E_2)} = T(f_0, f_1, f_2) : T(E_0, E_1, E_2) \rightarrow T(E'_0, E'_1, E'_2)$ , where  $E'_i = f^{-1}(E_i)$  and  $f_i = f|_{E'_i}$  for  $i = 0, 1, 2$ .

In addition, the restriction of  $\sigma_{E'}$  to  $T(E'_0, E'_1, E'_2)$  is the identity on the same component of  $\widehat{Tot}_{(0)}^*(E'_0, E'_1 + E'_2)$ .

On the other hand, the restriction of  $\sigma_E$  to  $T(E_0, E_1, E_2)$  is the identity on the same component of  $\widehat{Tot}_{(0)}^*(E_0, E_1 + E_2)$ , whereas the restriction of  $[\widehat{Tot}(\widehat{Tot}_{(0)}^*(T))](f)$  to  $\widehat{Tot}_{(0)}^*(E_0, E_1 + E_2)$  is  $\widehat{Tot}_{(0)}^*(f_0, f_1 + f_2)$ , since  $f|_{f^{-1}(E_0+E_1)} = f_0 + f_1$ .

Finally,  $\widehat{Tot}_{(0)}^*(f_0, f_1 + f_2)|_{T(E_0, E_1, E_2)}$  coincides with  $T(f_0, f_1, f_2)$  by definition, and the proof is concluded.  $\square$

The above proposition also holds for the functor  $Tot$ .

**(1.4.10)** Consider  $T \in 3 - \Delta^\circ \mathcal{D}$  and  $n \geq -1$ . We define

$$Tot^*(T([n], -, -)) = \begin{cases} Tot^+(T([n], -, -)) & \text{if } n > -1 \\ Tot(T([-1], -, -)) & \text{if } n = -1 \end{cases}.$$

Similarly,

$$Tot^*(T(-, -, [n])) = \begin{cases} Tot^+(T(-, -, [n])) & \text{if } n > -1 \\ Tot(T(-, -, [-1])) & \text{if } n = -1 \end{cases}.$$

**COROLLARY 1.4.11.** *There exists functors  $Tot_{(0)}, Tot_{(2)} : 3 - \Delta^\circ \mathcal{D} \rightarrow 2 - \Delta^\circ \mathcal{D}$  such that*

$$\begin{aligned} [Tot_{(0)}(T)]([n], [m]) &= [\widehat{Tot}^*(T([n], -, -))](m) \quad \text{and} \\ [Tot_{(2)}(T)]([n], [m]) &= [\widehat{Tot}^*(T(-, -, [m]))](n). \end{aligned}$$

Moreover, the functors  $Tot_3, Tot \circ Tot_{(0)}$  and  $Tot \circ Tot_{(2)} : 3 - \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  are isomorphic.

*Proof.* We have that  $Tot_{(0)} = \mathcal{I} \circ \widehat{Tot}_{(0)} \circ \mathcal{P}$ , and the same holds for  $Tot_{(2)}$ . More specifically, given  $T \in 3 - \Delta^\circ \mathcal{D}$  then

$$[\mathcal{I} \circ \widehat{Tot}_{(0)} \circ \mathcal{P}(T)]([n], [m]) = [\widehat{Tot}_{(0)} \circ \mathcal{P}(T)]([n], [m]) = [\widehat{Tot}^*((\mathcal{P}T)([n], -, -))]( [m] ),$$

that agrees with  $[Tot^*(T([n], -, -))]( [m] )$  since  $Tot$  and  $Tot^+$  are obtained from  $\widehat{Tot}$  and  $\widehat{Tot}^+$  by composition with  $\mathcal{I}$  and  $\mathcal{P}$ .

Consequently  $Tot_{(0)}$  and  $Tot_{(1)}$  are functors, and the statement follows from 1.4.9 together with  $\mathcal{P}\mathcal{I} \simeq Id$ .  $\square$

## 1.5 Simplicial cylinder object

In this section we introduce the *simplicial cylinder object*, that is a generalization of the simplicial cone object  $Cf$  associated with a morphism  $f : X \rightarrow Y$  between simplicial objects in  $\mathcal{D}$ , 1.3.17.

**DEFINITION 1.5.1.** Let  $\Omega(\mathcal{D})$  be the category of pairs  $(f, \epsilon)$  consisting of diagrams in  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \epsilon \downarrow & & \\ X_{-1} \times \Delta & & \end{array}$$

A morphism in  $\Omega(\mathcal{D})$  is a triple  $(\alpha, \beta, \gamma) : (f, \epsilon) \rightarrow (f', \epsilon')$  such that

$$\begin{array}{ccccc} X_{-1} \times \Delta & \xleftarrow{\epsilon} & X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ X'_{-1} \times \Delta & \xleftarrow{\epsilon'} & X' & \xrightarrow{f'} & Y' \end{array}$$

commutes.

In a similar way, we will denote by  $\text{Co}\Omega(\mathcal{D})$  the category whose objects are the diagrams in  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccc} X & \xleftarrow{f} & Y \\ \epsilon \uparrow & & \\ X_{-1} \times \Delta & & \end{array}$$

**(1.5.2)** Let  $\psi : \Omega(\mathcal{D}) \rightarrow 2 - \Delta_+^\circ \mathcal{D}$  be the functor that maps the pair  $(f, \epsilon)$  into the biaugmented bisimplicial (1.6) of 1.3.5.

**DEFINITION 1.5.3** (Simplicial cylinder object).

The simplicial cylinder functor  $Cyl : \Omega(\mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  is the composition

$$\Omega(\mathcal{D}) \xrightarrow{\psi} 2 - \Delta_+^\circ \mathcal{D} \xrightarrow{Tot} \Delta^\circ \mathcal{D} .$$

In other words,  $Cyl(f, \epsilon) = Tot(\psi(f, \epsilon))$ .

(1.5.4) Having in mind the description 1.3.10 of  $Tot$ , the functor  $Cyl$  can be described as follows.

Denote by  $u_i : [n] \rightarrow [1]$  the morphisms in  $\Delta$  with  $u_i([n]) = i$ , if  $i = 0, 1$  and

$$\Lambda_n = \{\sigma : [n] \rightarrow [1] \mid \sigma \neq u_0, u_1\} .$$

Given  $\sigma : [n] \rightarrow [1]$ , we will identify the ordered set  $\sigma^{-1}(i)$  with the corresponding object in  $\Delta_+$  for  $i = 0, 1$ . Then

$$[Cyl(f, \epsilon)]([n]) = \coprod_{\sigma : [n] \rightarrow [1]} Cyl(f, \epsilon)^{(\sigma)} ,$$

where

$$[Cyl(f, \epsilon)]^{(\sigma)} = \begin{cases} Y([n]) & \text{if } \sigma = u_1 \\ X_{-1} & \text{if } \sigma = u_0 \\ X(\sigma^{-1}(1)) & \text{if } \sigma \in \Lambda_n . \end{cases}$$

If  $\theta : [m] \rightarrow [n]$ , the restriction of  $\Theta = [Cyl(f, \epsilon)](\theta)$  to the component indexed by  $(\sigma)$  is

$$\Theta|_{(\sigma)} = \begin{cases} X(\theta|_{(\sigma\theta)^{-1}(1)}) : X(\sigma^{-1}(1)) \rightarrow X((\sigma\theta)^{-1}(1)) & \text{if } \sigma\theta \in \Lambda_m \\ Y(\theta) : Y([n]) \rightarrow Y([m]) & \text{if } \sigma = u_1 \\ f([m]) \circ X(\theta|_{(\sigma\theta)^{-1}(1)}) : X(\sigma^{-1}(1)) \rightarrow Y([m]) & \text{if } \sigma \in \Lambda_n \text{ and } \sigma\theta = u_1 \\ Id : X_{-1} \rightarrow X_{-1} & \text{if } \sigma = u_0 \\ \epsilon(\sigma^{-1}(1)) : X(\sigma^{-1}(1)) \rightarrow X_{-1} & \text{if } \sigma \in \Lambda_n \text{ and } \sigma\theta = u_0 . \end{cases} \quad (1.7)$$

(1.5.5) More specifically,  $Cyl(f, \epsilon)$  is in degree  $n$

$$Cyl(f, \epsilon)_n = Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} .$$

The face maps  $d_i : Cyl(f, \epsilon)_n \rightarrow Cyl(f, \epsilon)_{n-1}$  are given componentwise by  $d_i|_{Y_n} = d_i^Y$ , and if  $1 \leq k \leq n+1$  then

$$d_i|_{X_{n-k}} = \begin{cases} d_{i-k}^X & \text{if } i \geq k \\ Id_{X_{n-k}} & \text{if } i < k \text{ and } (k, i) \neq (1, 0) \\ f_{n-1} & \text{if } (k, i) = (1, 0) \end{cases}$$



where  $d_0^X = \epsilon_0 : X_0 \rightarrow X_{-1}$ . Visually, if  $1 \leq i \leq n$ , then  $d_i$  is

$$\begin{array}{ccccccc} Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup \cdots \sqcup X_{n-i} \sqcup X_{n-i-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \\ \downarrow d_i^Y \quad \downarrow d_{i-1}^X \quad \downarrow d_{i-2}^X \quad \quad \quad \downarrow d_0^X \quad \downarrow Id \quad \quad \quad \downarrow Id \quad \downarrow Id \\ Y_{n-1} \sqcup X_{n-2} \sqcup X_{n-3} \sqcup \cdots \sqcup X_{n-i-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \end{array}$$

whereas  $d_0$  is

$$\begin{array}{ccccccc} Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup X_{n-3} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \\ \downarrow d_0^Y \quad \downarrow f_{n-1} \quad \downarrow Id \quad \downarrow Id \quad \quad \quad \downarrow Id \quad \downarrow Id \\ Y_{n-1} \sqcup X_{n-2} \sqcup X_{n-3} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \end{array}$$

The degeneracy maps  $s_j : Cyl(f, \epsilon)_n \rightarrow Cyl(f, \epsilon)_{n+1}$  are defined as  $s_j|_{Y_n} = s_j^Y$  and given  $1 \leq k \leq n+1$  then

$$s_j|_{X_{n-k}} = \begin{cases} s_{j-k}^X & \text{if } j \geq k \\ Id_{X_{n-k}} & \text{if } j < k, \end{cases}$$

that is to say

$$\begin{array}{ccccccc} Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup \cdots \sqcup X_{n-j} \sqcup X_{n-j-1} \sqcup \cdots \sqcup X_{-1} \\ \downarrow s_j^Y \quad \downarrow s_{j-1}^X \quad \downarrow s_{j-2}^X \quad \quad \quad \downarrow s_0^X \quad \downarrow Id \quad \quad \quad \downarrow Id \\ Y_{n+1} \sqcup X_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_{n-j+1} \sqcup X_{n-j} \sqcup X_{n-j-1} \sqcup \cdots \sqcup X_{-1} \end{array}$$

**(1.5.6)** Given  $D : X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  in  $\Omega(\mathcal{D})$ , it follows from 1.3.7 that the canonical morphisms

$$i_{Y_n} : Y_n \rightarrow Cyl(D)_n \text{ and } i_{X_{-1}} : X_{-1} \rightarrow Cyl(D)_n$$

induce the following diagram, natural in  $(f, \epsilon)$ ,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \epsilon \downarrow & & \downarrow i_Y \\ X_{-1} \times \Delta & \xrightarrow{i_{X_{-1}}} & Cyl(D) \end{array} \quad (1.8)$$

Note that if 0 is an initial object in  $\mathcal{D}$ , the morphisms  $i_{X_{-1}}$  and  $i_Y$  are just the image under the functor  $Cyl$  of the maps

$$\begin{array}{ccc} X_{-1} \times \Delta & \xleftarrow{\quad} 0 & \xrightarrow{\quad} 0 \\ Id \downarrow & & \downarrow \\ X_{-1} \times \Delta & \xleftarrow{\epsilon} X & \xrightarrow{f} Y \end{array} \quad ; \quad \begin{array}{ccc} 0 & \xleftarrow{\quad} 0 & \xrightarrow{\quad} Y \\ \downarrow & & \downarrow \\ X_{-1} \times \Delta & \xleftarrow{\epsilon} X & \xrightarrow{f} Y \end{array}$$

since  $Cyl(X_{-1} \times \Delta \leftarrow 0 \rightarrow 0) = X_{-1} \times \Delta$  and  $Cyl(0 \leftarrow 0 \rightarrow Y) = Y$ .

**REMARK 1.5.7.** Let  $Fl(\Delta^\circ \mathcal{D})$  be the category of morphisms in  $\Delta^\circ \mathcal{D}$ .

If  $\mathcal{D}$  has a final object 1, consider the inclusion  $\mathcal{I} : Fl(\Delta^\circ \mathcal{D}) \rightarrow \Omega(\mathcal{D})$  that maps the diagram  $1 \leftarrow X \xrightarrow{f} Y$  into the morphism  $f : X \rightarrow Y$ .

Hence  $\mathcal{I}$  is right adjoint to the forgetful functor  $U : \Omega(\mathcal{D}) \rightarrow Fl(\Delta^\circ \mathcal{D})$ ,  $U(X, f, \epsilon) = f$ , and the *simplicial cone* functor  $C : Fl(\Delta^\circ \mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  is  $Cyl \circ \mathcal{I}$ .

Given  $X \in \Delta^\circ \mathcal{D}$ ,  $CX$  will mean  $C(Id_X)$ , and if  $S$  is an object in  $\mathcal{D}$ ,  $Cyl(S)$  will denote  $Cyl(S \times \Delta, Id_{S \times \Delta}, Id_{S \times \Delta})$ .

Next we study some of the properties of the functor  $Cyl$ .

**PROPOSITION 1.5.8.** *The functor  $Cyl : \Omega(\mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  commutes with coproducts, that is*

$$Cyl(f, \epsilon) \sqcup Cyl(f', \epsilon') \simeq Cyl(f \sqcup f', \epsilon \sqcup \epsilon') .$$

*Proof.* The statement follows directly from the commutativity of the functors  $\Psi$  and  $Tot$  with coproducts (see 1.3.15).  $\square$

The proof of the following proposition will be given later in 1.7.15.

**PROPOSITION 1.5.9.** *For every  $D : X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  in  $\Omega(\mathcal{D})$ , the diagram (1.8) commutes up to simplicial homotopy, natural in  $D$ .*

**PROPOSITION 1.5.10.** *Given a commutative diagram in  $\Delta^\circ \mathcal{D}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \epsilon \downarrow & & \downarrow \rho' \\ X_{-1} \times \Delta & \xrightarrow{\rho} & T \times \Delta , \end{array} \quad (1.9)$$

*there exists a unique  $H : Cyl(f, \epsilon) \rightarrow T \times \Delta$  such that  $H \circ i_{X_{-1}} = \rho$  and  $H \circ i_Y = \rho'$ . Equivalently,*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ \epsilon \downarrow & & \downarrow i_Y & \searrow \rho' & \\ X_{-1} \times \Delta & \xrightarrow{i_{X_{-1}}} & Cyl(f, \epsilon) & \xrightarrow{\#} & T \times \Delta \\ & \searrow \rho & \downarrow \# & \nearrow H & \\ & & & & \end{array}$$

*In addition,  $H$  is natural in (1.9).*

*Proof.* The data  $T$ ,  $\rho$  and  $\rho'$  allows to construct  ${}_+Z \in (\Delta_+ \times \Delta_+)^{\circ}\mathcal{D}$  from  $\Psi(f, \epsilon)$ . The restriction of  ${}_+Z$  to  $2 - \Delta_+^{\circ}\mathcal{D}$  is  $\Psi(f, \epsilon)$  and  ${}_+Z_{-1,-1} = T$ .

Hence, the morphism  $H_0 : Y_0 \sqcup X_{-1} \rightarrow T$  with  $H_0|_{Y_0} = \rho'_0$ ,  $H_0|_{X_{-1}} = \rho$  is an augmentation of  $Cyl(f, \epsilon)$  by 1.3.8.

Moreover  $H_0$  is the unique morphism such that  $H_0 \circ i_{X_{-1}} = \rho_0$  and  $H_0 \circ i_{Y_0} = \rho'_0$ . We deduce from 1.2.6 that  $H_0$  induces  $H : Cyl(f, \epsilon) \rightarrow T \times \Delta$  with  $H_n = H_0 \circ (d_0)^n$ , and such that  $H \circ i_{X_{-1}} = \rho : X_{-1} \times \Delta \rightarrow T \times \Delta$  and  $H \circ i_Y = \rho' : Y \rightarrow T \times \Delta$ , because these morphisms agree in degree 0.

Since  $H$  is determined by  $H_0$ , it follows that  $H$  is the unique morphism satisfying the required equality, and  $H$  is natural in (1.9) because  $H_0$  is so.  $\square$

Now we develop a property of  $Cyl$  that will be needed later for the study of the relationship between simplicial descent categories and triangulated categories.

**(1.5.11)** We have that  $\Delta^{\circ}\mathcal{D}$  has coproducts because  $\mathcal{D}$  has. Then we can consider the cylinder functor of a morphism  $f : X \rightarrow Y$  between bisimplicial objects, where  $X$  has an augmentation  $\epsilon$ . This can be an augmentation with respect to any of the two simplicial indexes of  $X$ . Therefore we need to introduce the following notations.

**DEFINITION 1.5.12.** Consider the category  $\Omega^{(1)}(\Delta^{\circ}\mathcal{D})$  whose objects are the diagrams

$$\Delta \times Z_{-1} \xleftarrow{\epsilon} Z \xrightarrow{f} T ,$$

that in degree  $n, m$  is  $Z_{-1,m} \xleftarrow{\epsilon_{n,m}} Z_{n,m} \xrightarrow{f_{n,m}} T_{n,m}$ .

Hence, the functor  $Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)} : \Omega^{(1)}(\Delta^{\circ}\mathcal{D}) \rightarrow \Delta^{\circ}\Delta^{\circ}\mathcal{D}$  is

$$Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)_{n,m} = T_{n,m} \sqcup Z_{n-1,m} \sqcup \cdots \sqcup Z_{0,m} \sqcup Z_{-1,m} .$$

If  $\alpha : [m'] \rightarrow [m]$  then  $[Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)](Id, \alpha) : Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)_{n,m} \rightarrow Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)_{n,m'}$  is

$$T(Id, \alpha) \sqcup Z(Id, \alpha) \sqcup \cdots \sqcup Z(Id, \alpha) \sqcup Z_{-1}(\alpha) ,$$

whereas if  $\beta : [n'] \rightarrow [n]$ , we define  $[Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)](\beta, Id) : Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)_{n,m} \rightarrow Cyl_{\Delta^{\circ}\mathcal{D}}^{(1)}(f, \epsilon)_{n',m}$  using the formulae (1.7), by forgetting the index  $m$ .

The category  $\Omega^{(2)}(\Delta^{\circ}\mathcal{D})$  is defined in the same way, but considering the diagrams

$$D : Z_{-1} \times \Delta \xleftarrow{\epsilon} Z \xrightarrow{f} T .$$

Similarly, we define the diagram  $Cyl_{\Delta^\circ \mathcal{D}}^{(2)} : \Omega^{(2)}(\Delta^\circ \mathcal{D}) \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}$  by applying  $Cyl$  to the second index of the diagram  $D$ . Then, we obtain the following square of functors

$$\begin{array}{ccc} \Omega^{(1)}(\Delta^\circ \mathcal{D}) & \xrightarrow{Cyl_{\Delta^\circ \mathcal{D}}^{(1)}} & \Delta^\circ \Delta^\circ \mathcal{D} \\ \Gamma \downarrow & & \downarrow \Gamma \\ \Omega^{(2)}(\Delta^\circ \mathcal{D}) & \xrightarrow{Cyl_{\Delta^\circ \mathcal{D}}^{(2)}} & \Delta^\circ \Delta^\circ \mathcal{D} \end{array} \quad (1.10)$$

**(1.5.13)** As happens with  $Cyl$ , we have the canonical inclusions of  $\Delta \times Z_{-1}$  (resp.  $Z_{-1} \times \Delta$ ) and  $T$  in  $Cyl_{\Delta^\circ \mathcal{D}}^{(1)}$  (resp.  $Cyl_{\Delta^\circ \mathcal{D}}^{(2)}$ ).

**(1.5.14)** Assume that the following diagram commutes in  $\mathcal{D}$

$$\begin{array}{ccccc} Z' & \xleftarrow{g'} & X' & \xrightarrow{f'} & Y' \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow \\ Z & \xleftarrow{g} & X & \xrightarrow{f} & Y \\ \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow \\ Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y'' \end{array} \quad (1.11)$$

Consider (1.11) in  $\Delta^\circ \mathcal{D}$  through the functor  $- \times \Delta$ . Applying  $Cyl$  by rows and columns we obtain

$$\begin{array}{ccc} Cyl(f', g') & \xleftarrow{\rho} & Cyl(f, g) \xrightarrow{\rho'} Cyl(f'', g'') \\ Cyl(\alpha', \alpha) & \xleftarrow{G} & Cyl(\beta', \beta) \xrightarrow{F} Cyl(\gamma', \gamma). \end{array}$$

Then the diagrams of  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccccc} Y' & \xleftarrow{\gamma} & Y & \xrightarrow{\gamma'} & Y'' \\ i \downarrow & & i \downarrow & & i \downarrow \\ Cyl(f', g') & \xleftarrow{\rho} & Cyl(f, g) & \xrightarrow{\rho'} & Cyl(f'', g'') \end{array} \quad ; \quad \begin{array}{ccccc} Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y'' \\ i \downarrow & & i \downarrow & & i \downarrow \\ Cyl(\alpha', \alpha) & \xleftarrow{G} & Cyl(\beta', \beta) & \xrightarrow{F} & Cyl(\gamma', \gamma). \end{array}$$

give rise to the morphisms between bisimplicial objects

$$\begin{array}{l} \varphi : \Delta \times Cyl(\gamma', \gamma) \rightarrow Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta) \\ \phi : Cyl(f'', g'') \times \Delta \rightarrow Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G). \end{array}$$

**LEMMA 1.5.15.** *Under the above notations, there exists a canonical morphism  $\Theta : Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G) \rightarrow Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta)$  in  $\Delta^\circ \Delta^\circ \mathcal{D}$ , such that the following diagram commutes*

$$\begin{array}{ccccc}
 & & Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G) & & \\
 & \nearrow \phi & \downarrow \Theta & \nwarrow i & \\
 Cyl(f'', g'') \times \Delta & & & & \Delta \times Cyl(\gamma', \gamma) \\
 & \searrow i & & \nearrow \varphi & \\
 & & Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta) & & 
 \end{array}$$

*Proof.* Set  $A.. = A \times \Delta \times \Delta \in \Delta^\circ \Delta^\circ \mathcal{D}$  if  $A \in \mathcal{D}$ , as well as  $h.. = h \times \Delta \times \Delta$  if  $h$  is a morphism in  $\mathcal{D}$ . The diagram of  $\Delta^\circ \Delta^\circ \mathcal{D}$

$$\begin{array}{ccccccc}
 Z'.. & \xleftarrow{g'..} & X'.. & \xrightarrow{f'..} & Y'.. & \xrightarrow{i} & Cyl(f', g') \times \Delta \\
 \alpha'.. \uparrow & & \beta'.. \uparrow & & \gamma'.. \uparrow & & \rho \times \Delta \uparrow \\
 Z.. & \xleftarrow{g..} & X.. & \xrightarrow{f..} & Y.. & \xrightarrow{i} & Cyl(f, g) \times \Delta \\
 \alpha'.. \downarrow & & \beta'.. \downarrow & & \gamma'.. \downarrow & & \rho' \times \Delta \downarrow \\
 Z''.. & \xleftarrow{g''..} & X''.. & \xrightarrow{f''..} & Y''.. & \xrightarrow{i} & Cyl(f'', g'') \times \Delta \\
 i \downarrow & & i \downarrow & & i \downarrow & & \phi \downarrow \\
 \Delta \times Cyl(\alpha', \alpha) & \xleftarrow{\Delta \times G} & \Delta \times Cyl(\beta', \beta) & \xrightarrow{\Delta \times F} & \Delta \times Cyl(\gamma', \gamma) & \xrightarrow{i} & Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G),
 \end{array} \tag{1.12}$$

where each  $i$  is degree-wise the canonical inclusion given by the coproduct, is commutative. (1.12) is in degrees  $n, m$

$$\begin{array}{ccccccc}
 Z' & \xleftarrow{g'} & X' & \xrightarrow{f'} & Y' & \xrightarrow{i_n} & Y' \sqcup \coprod^n X' \sqcup Z' \\
 \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \rho_n \uparrow \\
 Z & \xleftarrow{g} & X & \xrightarrow{f} & Y & \xrightarrow{i_n} & Y \sqcup \coprod^n X \sqcup Z \\
 \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow & & \rho'_n \downarrow \\
 Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y'' & \xrightarrow{i_n} & Y'' \sqcup \coprod^n X'' \sqcup Z'' \\
 i_m \downarrow & & i_m \downarrow & & i_m \downarrow & & \phi_{n,m} \downarrow \\
 Z'' \sqcup \coprod^m Z \sqcup Z' & \xleftarrow{G_m} & X'' \sqcup \coprod^m X \sqcup X' & \xrightarrow{F_m} & Y'' \sqcup \coprod^m Y \sqcup Y' & \xrightarrow{i} & T_{n,m}
 \end{array}$$

where  $T_{n,m} = (Y'' \sqcup \coprod^m Y \sqcup Y') \sqcup \coprod^n (X'' \sqcup \coprod^m X \sqcup X') \sqcup (Z'' \sqcup \coprod^m Z \sqcup Z')$ .

On the other hand, if  $R = Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta)$ , we have that

$R_{n,m} = (Y'' \sqcup \coprod^n X'' \sqcup Z'') \sqcup \coprod_m (Y \sqcup \coprod^n X \sqcup Z) \sqcup (Y' \sqcup \coprod^n X' \sqcup Z')$ , that is

obtained by reordering the coproduct in  $T_{n,m}$ .

Therefore, let  $\Theta_{n,m} : T_{n,m} \rightarrow R_{n,m}$  be the canonical isomorphism that reorders the coproduct.

It is clear that

$$\begin{array}{ccc}
 & \nearrow \phi_{n,m} & \\
 Y'' \sqcup \coprod^n X'' \sqcup Z'' & & (Y'' \sqcup \coprod^m Y \sqcup Y') \sqcup \coprod^n (X'' \sqcup \coprod^m X \sqcup X') \sqcup (Z'' \sqcup \coprod^m Z \sqcup Z') \\
 & \searrow i_n & \downarrow \Theta_{n,m} \\
 & & (Y'' \sqcup \coprod^n X'' \sqcup Z'') \sqcup \coprod^m (Y \sqcup \coprod^n X \sqcup Z) \sqcup (Y' \sqcup \coprod^n X' \sqcup Z')
 \end{array}$$

is commutative, and similarly  $\Theta_{n,m} \circ i_m = \varphi_{n,m}$ .

Hence, it remains to show that  $\Theta = \{\Theta_{n,m}\}_{n,m}$  is a morphism of bisimplicial objects.

Following the terminology in 1.5.4, write  $T_{n,m} = Cyl_{\Delta \circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G)_{n,m}$  as

$$\begin{aligned}
 & Cyl(\gamma', \gamma)_m^{u_1} \sqcup \coprod_{\rho \in \Lambda_n} Cyl(\beta', \beta)_m^\rho \sqcup Cyl(\alpha', \alpha)_m^{u_0} = \\
 & (Y''^{u_1, u_1} \sqcup \coprod_{\sigma \in \Lambda_m} Y^{u_1, \sigma} \sqcup Y'^{u_1, u_0}) \sqcup \coprod_{\rho \in \Lambda_n} (X''^{\rho, u_1} \sqcup \coprod_{\sigma \in \Lambda_m} X^{\rho, \sigma} \sqcup X'^{\rho, u_0}) \sqcup (Z''^{u_0, u_1} \sqcup \coprod_{\sigma \in \Lambda_m} Z^{u_0, \sigma} \sqcup Z'^{u_0, u_0})
 \end{aligned}$$

On the other hand,  $R_{n,m} = Cyl_{\Delta \circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta)_{n,m}$  is

$$\begin{aligned}
 & Cyl(f'', g'')_n^{u_1} \sqcup \coprod_{\rho \in \Lambda_m} Cyl(f, g)_n^\rho \sqcup Cyl(f', g')_n^{u_0} = \\
 & (Y''^{u_1, u_1} \sqcup \coprod_{\sigma \in \Lambda_n} X''^{u_1, \sigma} \sqcup Z''^{u_1, u_0}) \sqcup \coprod_{\rho \in \Lambda_m} (Y^{\rho, u_1} \sqcup \coprod_{\sigma \in \Lambda_n} X^{\rho, \sigma} \sqcup Z^{\rho, u_0}) \sqcup (Y'^{u_0, u_1} \sqcup \coprod_{\sigma \in \Lambda_n} X'^{u_0, \sigma} \sqcup Z'^{u_0, u_0})
 \end{aligned}$$

Then,  $\Theta_{n,m}$  maps the component  $(\rho, \sigma)$  of  $T_{n,m}$  into the component  $(\sigma, \rho)$  of  $R_{n,m}$ .

If  $\theta : [n'] \rightarrow [n]$ , the verification of the equalities  $\Theta_{n',m} \circ T(\theta, Id) = R(\theta, Id) \circ \Theta_{n,m}$ , and  $\Theta_{m,n'} \circ T(Id, \theta) = R(Id, \theta) \circ \Theta_{m,n}$  is a straightforward computation.

Let us see, for instance, the first equality, because the second one is totally similar. We have that

$$T(\theta, Id)|_{(\rho, \sigma)} = \begin{cases} Id : Cyl(\beta', \beta)_m^\rho \rightarrow Cyl(\beta', \beta)_m^{\rho\theta} & \text{if } \rho\theta \in \Lambda_{n'} \\ F_m : Cyl(\beta', \beta)_m^\rho \rightarrow Cyl(\gamma', \gamma)_m^{u_1} & \text{if } \rho\theta = u_1 \text{ and } \rho \in \Lambda_n \\ G_m : Cyl(\beta', \beta)_m^\rho \rightarrow Cyl(\alpha', \alpha)_m^{u_0} & \text{if } \rho\theta = u_0 \text{ and } \rho \in \Lambda_n \\ Id : Cyl(\gamma', \gamma)_m^{u_1} \rightarrow Cyl(\gamma', \gamma)_m^{u_1} & \text{if } \rho = u_1 \\ Id : Cyl(\alpha', \alpha)_m^{u_0} \rightarrow Cyl(\alpha', \alpha)_m^{u_0} & \text{if } \rho = u_0 \end{cases}$$

Note also that the restriction of  $R(\theta, Id) \circ \Theta_{n,m}$  to the component  $(\rho, \sigma)$  agrees with the restriction of  $R(\theta, Id)$  to the component  $(\sigma, \rho)$ , that is by definition

$$R(\theta, Id)|_{(\sigma, \rho)} = \begin{cases} Cyl(f, g)(\theta)|_{(\sigma, \rho)} : Cyl(f, g)_n^\sigma \rightarrow Cyl(f, g)_{n'}^\sigma & \text{if } \sigma \in \Lambda_m \\ Cyl(f'', g'')(\theta)|_{(\sigma, \rho)} : Cyl(f'', g'')_n^{u_1} \rightarrow Cyl(f'', g'')_{n'}^{u_1} & \text{if } \sigma = u_1 \\ Cyl(f', g')(\theta)|_{(\sigma, \rho)} : Cyl(f', g')_n^{u_0} \rightarrow Cyl(f', g')_{n'}^{u_0} & \text{if } \sigma = u_0 . \end{cases}$$

We remind that

$$Cyl(f, g)(\theta)|_{(\sigma, \rho)} = \begin{cases} Id : X^{\sigma, \rho} \rightarrow X^{\sigma, \rho\theta} & \text{if } \rho\theta \in \Lambda_{n'} \\ Id : Y^{\sigma, u_1} \rightarrow X^{\sigma, u_1} & \text{if } \rho = u_1 \\ Id : Z^{\sigma, u_0} \rightarrow X^{\sigma, u_0} & \text{if } \rho = u_0 \\ f : X^{\sigma, \rho} \rightarrow Y^{\sigma, u_1} & \text{if } \rho \in \Lambda_n \text{ and } \rho\theta = u_1 \\ g : X^{\sigma, \rho} \rightarrow Z^{\sigma, u_0} & \text{if } \rho \in \Lambda_n \text{ and } \rho\theta = u_0 , \end{cases}$$

and analogously for  $Cyl(f'', g'')(\theta)|_{(\sigma, \rho)}$  and  $Cyl(f', g')(\theta)|_{(\sigma, \rho)}$ .

Assume that  $\rho\theta \in \Lambda_{n'}$ . Then

$$T(\theta, Id)|_{(\rho, \sigma)} = \begin{cases} Id : X''^{\rho, u_1} \rightarrow X''^{\rho\theta, u_1} & \text{if } \sigma = u_1 \\ Id : X^{\rho, \sigma} \rightarrow X^{\rho\theta, \sigma} & \text{if } \sigma \in \Lambda_m \\ Id : X'^{\rho, u_0} \rightarrow X'^{\rho\theta, u_0} & \text{if } \sigma = u_0 . \end{cases}$$

Hence, the result of interchanging the indexes in the above formula is

$$\Theta_{n,m} T(\theta, Id)|_{(\rho, \sigma)} = \left\{ \begin{array}{ll} Id : X''^{u_1, \rho} \rightarrow X''^{u_1, \rho\theta} & \text{if } \sigma = u_1 \\ Id : X^{\sigma, \rho} \rightarrow X^{\sigma, \rho\theta} & \text{if } \sigma \in \Lambda_m \\ Id : X'^{u_0, \rho} \rightarrow X'^{u_0, \rho\theta} & \text{if } \sigma = u_0 \end{array} \right\} = R(\theta, Id)|_{(\rho, \sigma)} ,$$

and the equality holds as in the remaining cases.

□

**DEFINITION 1.5.16** (Simplicial path object).

The simplicial path functor  $Path : Co\Omega(\mathcal{D}) \rightarrow \Delta\mathcal{D}$  is just the dual notion of  $Cyl$ , consequently it satisfies the dual properties included in this section.

## 1.6 Symmetric notions of cylinder and cone

The biaugmented bisimplicial object  $Z$  associated with an object  $(f, \epsilon)$  in  $\Omega(\mathcal{D})$  is clearly asymmetric, and we could consider as well the object  $Z'$  obtained from

$Z$  by interchanging the indexes. The total simplicial object of  $Z'$  is another cylinder object associated with  $(f, \epsilon)$  that will be studied in this section.

**(1.6.1)** Let  $op : \Delta \rightarrow \Delta$  be the isomorphism of categories that ‘reverses the order’, introduced in 1.1.5. Denote by  $\Upsilon : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  the functor obtained by composition with  $op$ .

Therefore

$$(\Upsilon X)_n = X_n \quad d_i^{\Upsilon X} = d_{n-i}^X : X_n \rightarrow X_{n-1} \quad s_j^{\Upsilon X} = s_{n-j}^X : X_n \rightarrow X_{n+1}.$$

Let  $\Upsilon : \Omega(\mathcal{D}) \rightarrow \Omega(\mathcal{D})$  be the induced functor, that is also an isomorphism, and is given by  $\Upsilon(f, \epsilon) = (\Upsilon(f), \Upsilon(\epsilon))$ .

The result of “conjugate” the cylinder functor  $Cyl$  with respect to  $\Upsilon$  is the following alternative definition of cylinder.

**DEFINITION 1.6.2.** Set  $Cyl' = \Upsilon \circ Cyl \circ \Upsilon : \Omega(\mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$ .

Given  $X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  in  $\Omega(\mathcal{D})$ , then  $Cyl'(D)$  is in degree  $n$

$$Cyl'(D)_n = Y_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1}.$$

The face morphisms  $d_i^{Cyl'(D)} : Cyl'(D)_n \rightarrow Cyl'(D)_{n-1}$  are defined as

$$d_i^{Cyl'(D)}|_{Y_n} = d_i^Y, \quad d_i^{Cyl'(D)}|_{X_k} = \begin{cases} d_i^X & i \leq k \\ Id & i > k \text{ } (i, k) \neq (n, n-1) \\ f_{n-1} & (i, k) = (n, n-1) \end{cases}$$

The degeneracy maps  $s_j^{Cyl'(D)} : Cyl'(D)_n \rightarrow Cyl'(D)_{n+1}$  are

$$s_j^{Cyl'(D)}|_{Y_n} = s_j^Y, \quad s_j^{Cyl'(D)}|_{X_k} = \begin{cases} s_j^X & j \leq k \\ Id & j > k \end{cases}.$$

Visually, for  $0 \leq i < n$ ,  $d_i^{Cyl'(D)}$  is

$$\begin{array}{ccccccc} Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup \cdots & \sqcup & X_i \sqcup X_{i-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \\ \partial_i^Y \searrow & \partial_i^X \searrow & \partial_i^X \searrow & \partial_i^X \searrow & \downarrow Id & \downarrow Id & \downarrow Id \\ Y_{n-1} \sqcup X_{n-2} \sqcup X_{n-3} \sqcup \cdots & \sqcup & X_{i-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \end{array}$$

The case  $i = n$  is

$$\begin{array}{ccccccc} Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup X_{n-2} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \\ \partial_n^Y \searrow & \downarrow f_n & \downarrow Id & \downarrow Id & \downarrow Id & \downarrow Id \\ Y_{n-1} \sqcup X_{n-2} \sqcup X_{n-2} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \end{array}$$



Finally  $s_j^{Cyl'(D)}$  is expressed as

$$\begin{array}{ccccccc}
 & Y_n \sqcup X_{n-1} \sqcup X_{n-2} \sqcup \cdots \sqcup X_j \sqcup X_{j-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \\
 s_j^Y \swarrow & s_j^X \swarrow & s_j^X \swarrow & & s_j^X \swarrow & \downarrow Id & \downarrow Id & \downarrow Id \\
 Y_{n+1} \sqcup X_n \sqcup X_{n-1} \sqcup \cdots & X_{j+1} \sqcup X_j \sqcup X_{j-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} .
 \end{array}$$

(1.6.3) Again, it follows from the properties of  $Tot$  the existence of canonical inclusions

$$X_{-1} \times \Delta \rightarrow Cyl'(f, \epsilon) \quad Y \rightarrow Cyl'(f, \epsilon) .$$

The next result is a consequence of the definitions of  $Tot$  and  $Cyl'$ . We denote by

$$\Gamma : 2 - \Delta_+ \longrightarrow 2 - \Delta_+$$

the functor that interchanges the indexes of a biaugmented bisimplicial object.

**PROPOSITION 1.6.4.** *The functor  $Cyl' : \Omega(\mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  agrees with the composition*

$$\Omega(\mathcal{D}) \xrightarrow{\psi} 2 - \Delta_+ \xrightarrow{\Gamma} 2 - \Delta_+ \xrightarrow{Tot} \Delta^\circ \mathcal{D} ,$$

where  $\psi$  is the functor given in (1.5.2).

In other words, the simplicial object  $Cyl'(f, \epsilon)$  coincides with the total object of the biaugmented bisimplicial object obtained by interchanging the indexes in (1.6), that is to say

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 X_{-1} \longleftarrow X_0 \rightleftharpoons X_1 \rightleftharpoons X_2 \rightleftharpoons X_3 & \cdots & & & & & \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & & \\
 X_{-1} \longleftarrow X_0 \rightleftharpoons X_1 \rightleftharpoons X_2 \rightleftharpoons X_3 & \cdots & & & & & \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & & \\
 X_{-1} \longleftarrow X_0 \rightleftharpoons X_1 \rightleftharpoons X_2 \rightleftharpoons X_3 & \cdots & & & & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & Y_0 \rightleftharpoons Y_1 \rightleftharpoons Y_2 \rightleftharpoons Y_3 & \cdots & & & & 
 \end{array}$$

**REMARK 1.6.5.** The functor  $Cyl'$  satisfies as well the analogous propositions to 1.5.8, 1.5.9 and 1.5.10, that we will labelled as 1.5.8', 1.5.9' and 1.5.10'.

**DEFINITION 1.6.6.** If  $\mathcal{D}$  has a final object, the following alternative notion of simplicial cone is induced by  $Cyl'$

$$C' = \Upsilon \circ C \circ \Upsilon : Fl(\Delta^\circ \mathcal{D}) \rightarrow \Delta^\circ \mathcal{D} .$$

The following proposition will be useful in the next chapter, and it will be a key point in the study of the relationship between the cone and cylinder axioms.

**(1.6.7)** Consider a commutative diagram in  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccccc} & & X_{-1} \times \Delta & \xrightarrow{h} & U_{-1} \times \Delta \\ & \nearrow r & \uparrow \beta & & \uparrow \gamma \\ Y_{-1} \times \Delta & & X & \xrightarrow{g} & U \\ \uparrow \alpha & \searrow q & & & \searrow p \\ Y & \xrightarrow{f} & V & & \end{array} , \quad (1.13)$$

We will denote by

$$U_{-1} \times \Delta \xleftarrow{\delta} Cyl'(g, \beta) \xrightarrow{t} Cyl'(f, \alpha)$$

the object of  $\Omega(\mathcal{D})$  obtained by applying  $Cyl'$  in one direction, and by

$$Y_{-1} \times \Delta \xleftarrow{\zeta} Cyl(q, \beta) \xrightarrow{u} Cyl(p, \gamma) .$$

the result of applying  $Cyl$  to (1.13) in the other sense.

**PROPOSITION 1.6.8.** *There exists a natural isomorphism in (1.13)*

$$Cyl'(u, \zeta) \simeq Cyl(t, \delta) .$$

*Proof.* We can add in a suitable way two new simplicial indexes to each simplicial object in (1.13) in order to obtain a 3-augmented 3-simplicial object  $T$  (see 1.4.2). In other words, for  $i, j, k \geq 0$  define

$$\begin{array}{llll} T_{i,j,k} = X_j & T_{i,j,-1} = U_j & T_{-1,j,k} = Y_j & T_{-1,j,-1} = V_j \\ T_{i,-1,k} = X_{-1} & T_{i,-1,-1} = U_{-1} & T_{-1,-1,k} = Y_{-1} & \end{array}$$

It follows from 1.4.11 that  $Tot \circ Tot_{(0)}(T) \simeq Tot \circ Tot_{(2)}(T)$ .

If  $i \geq 0$ , we can fix as  $i$  the first index of  $T$  and apply  $Tot^+$ . The result is

the augmented simplicial object  $Cyl'(g, \beta) \rightarrow U_{-1} \times \Delta$ , whereas if  $i = -1$  we obtain  $Cyl'(f, \alpha)$ . Hence

$$Tot_{(0)}(T)([n], [m]) = \begin{cases} Cyl'(g, \beta)_m & \text{if } n, m \geq 0 \\ U_{-1} & \text{if } m = -1 \\ Cyl'(f, \alpha) & \text{if } n = -1 . \end{cases}$$

Then,  $Tot_{(0)}(T)$  is the biaugmented bisimplicial object associated with

$$Cyl'(f, \alpha) \xleftarrow{t} Cyl'(g, \beta) \xrightarrow{\delta} U_{-1} \times \Delta ,$$

in (1.3.5), so  $Tot \circ Tot_{(0)}(T) = Cyl(t, \delta)$ .

On the other hand, if we fix  $k \geq 0$  in  $T$  and apply  $Tot^+$  we get  $Cyl(q, \beta) \rightarrow Y_{-1} \times \Delta$ , whereas setting  $k = -1$  and applying  $Tot$  we obtain  $Cyl(p, \gamma)$ . Consequently,

$$Tot_{(2)}(T)([n], [m]) = \begin{cases} Cyl(q, \beta)_n & \text{if } n, m \geq 0 \\ Y_{-1} & \text{if } n = -1 \\ Cyl(p, \gamma)_n & \text{if } m = -1 . \end{cases}$$

Therefore  $Tot \circ Tot_{(2)}(T) = Cyl'(u, \zeta)$ , and we are done.  $\square$

**COROLLARY 1.6.9.** *Let  $f : A \rightarrow B$  and  $g : A \rightarrow C$  morphisms in  $\mathcal{D}$ . Then*

$$Cyl(f \times \Delta, g \times \Delta) \simeq Cyl'(g \times \Delta, f \times \Delta) .$$

*Moreover, this isomorphism is compatible with the respective inclusion of  $B \times \Delta$  and  $C \times \Delta$  into both cylinder objects.*

*Proof.* Let  $0$  be the initial object of  $\mathcal{D}$ , and  $D.$  be the constant simplicial object  $D \times \Delta$ , for every object  $D$  in  $\mathcal{D}$ . It is enough to apply the above proposition to

$$\begin{array}{ccccc} & & A. & \xrightarrow{g.} & C. \\ & \swarrow f. & \uparrow & & \uparrow \\ B. & & 0. & \xrightarrow{\quad} & 0. \\ & \swarrow & \downarrow & & \downarrow \\ 0. & \xrightarrow{\quad} & 0. & & 0. \end{array} ,$$

and note that  $Cyl'(D. \leftarrow 0. \rightarrow 0.) = Cyl(D. \leftarrow 0. \rightarrow 0.) = D.$  for any  $D$  in  $\mathcal{D}$ . In addition, since the isomorphism obtained in this way is just to reorder a coproduct, it is clear that the canonical inclusions of  $B \times \Delta$  and  $C \times \Delta$  are preserved.  $\square$

## 1.7 Cubical cylinder object

The construction developed in this section is just a generalization of the cubical cylinder object  $\widetilde{Cyl}(X)$  associated with a simplicial object  $X$  in  $\mathcal{D}$ , 1.3.18.

**DEFINITION 1.7.1.** Let  $\square_1$  be the category

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \ .$$

Then, if we fix a category  $\mathcal{C}$ , the category  $\square_1 \mathcal{C}$  has as objects the diagrams in  $\mathcal{C}$

$$Z \xleftarrow{g} X \xrightarrow{f} Y \ ,$$

that will be represented by  $(f, g)$ .

The morphisms in  $\square_1 \mathcal{C}$  are commutative diagrams

$$\begin{array}{ccccc} Z & \longleftarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \longleftarrow & X' & \longrightarrow & Y' . \end{array}$$

Similarly,  $\square_1 \mathcal{C}$  is the category whose objects are diagrams

$$Z \xrightarrow{g} X \xleftarrow{f} Y \ .$$

**DEFINITION 1.7.2.** Define the functor  $\Phi : \square_1 \Delta^\circ \mathcal{D} \rightarrow 2 - \Delta_+^\circ \mathcal{D}$  as follows.

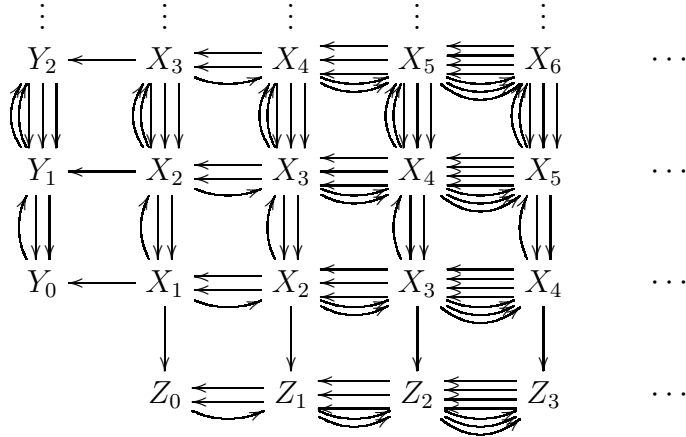
Given  $(f, g) \in \square_1 \Delta^\circ \mathcal{D}$ , the biaugmented bisimplicial object  $\Phi(f, g)$  is

$$T : \quad T_{-1, \cdot} \xleftarrow{\epsilon} T_{\cdot, \cdot}^+ \xrightarrow{\zeta} T_{\cdot, -1} \ ,$$

where  $T_{\cdot, \cdot}^+ = Dec(X)|_{\Delta \times \Delta}$  (see 1.3.18),  $T_{-1, \cdot} = Y$  and  $T_{\cdot, -1} = Z$ . In other words

$$T_{i,j} = \begin{cases} X_{i+j+1} & \text{if } i, j \geq 0 \\ Y_j & \text{if } i = -1 \\ Z_i & \text{if } j = -1 \end{cases} .$$

Visually,  $T$  can be visualized as



The horizontal augmentations,  $\epsilon_n : X_n \rightarrow Y_{n-1}$ , are equal to  $f_{n-1}d_0$ , whereas the vertical ones,  $\zeta_n : X_n \rightarrow Z_{n-1}$ , are  $g_{n-1}d_n$ .

**DEFINITION 1.7.3** (Cubical cylinder object).

We define the cubical cylinder functor  $\widetilde{Cyl} : \square_1^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  as the composition

$$\square_1^\circ \Delta^\circ \mathcal{D} \xrightarrow{\Phi} 2 - \Delta_+^\circ \mathcal{D} \xrightarrow{Tot} \Delta^\circ \mathcal{D} .$$

In other words,  $\widetilde{Cyl}(f, g) = Tot(\Phi(f, g))$ , that is in degree  $n$

$$\widetilde{Cyl}(f, g)_n = Y_n \sqcup X_n \sqcup \overset{n}{\cdots} \sqcup X_n \sqcup Z_n .$$

(1.7.4) Equivalently, consider the maps  $u_i : [n] \rightarrow [1]$  given by  $u_i([n]) = i$ ,  $i = 0, 1$ , and let  $\Lambda$  be the set of morphisms  $\sigma : [n] \rightarrow [1]$  different from  $u_0$  and  $u_1$ . Then

$$\widetilde{Cyl}(f, g)_n = \widetilde{Cyl}(f, g)_n^{u_1} \sqcup \coprod_{\sigma \in \Lambda} \widetilde{Cyl}(f, g)_n^\sigma \sqcup \widetilde{Cyl}(f, g)_n^{u_0} ,$$

where  $\widetilde{Cyl}(f, g)_n^{u_1} = Y_n$ ,  $\widetilde{Cyl}(f, g)_n^{u_0} = Z_n$  and  $\widetilde{Cyl}(f, g)_n^\sigma = X_n \forall \sigma \in \Lambda$ .

If  $\theta : [m] \rightarrow [n]$  is a morphism in  $\Delta$ , it follows that the restriction of  $\widetilde{Cyl}(f, g)(\theta) : \widetilde{Cyl}(f, g)_n \rightarrow \widetilde{Cyl}(f, g)_m$  to the component  $\sigma \in \Delta/[1]$  is

$$\widetilde{Cyl}(f, g)(\theta)|_{\widetilde{Cyl}(f, g)_n^\sigma} = \begin{cases} X(\theta) : X_n^\sigma \rightarrow X_m^{\sigma\theta} & \text{if } \sigma\theta \in \Lambda \\ f_m X(\theta) : X_n^\sigma \rightarrow Y_m^{u_1} & \text{if } \sigma \in \Lambda, \sigma\theta = u_1 \\ g_m X(\theta) : X_n^\sigma \rightarrow Z_m^{u_1} & \text{if } \sigma \in \Lambda, \sigma\theta = u_0 \\ Y(\theta) : Y_n^{u_1} \rightarrow Y_m^{u_1} & \text{if } \sigma = u_1 \\ Z(\theta) : Z_n^{u_1} \rightarrow Z_m^{u_1} & \text{if } \sigma = u_0 . \end{cases}$$

(1.7.5) Consider  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\square_1^\circ \Delta^\circ \mathcal{D}$ . The canonical morphisms

$$j_{Y_n} : Y_n \rightarrow \widetilde{Cyl}(f, g)_n \text{ and } j_{Z_n} : Z_n \rightarrow \widetilde{Cyl}(f, g)_n$$

give rise to the following diagram, natural in  $(f, g)$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow j_Y \\ Z & \xrightarrow{j_Z} & \widetilde{Cyl}(f, g) . \end{array} \quad (1.14)$$

**REMARK 1.7.6.** If  $X$  is a simplicial object in  $\mathcal{D}$ , then  $\widetilde{Cyl}(Id_X, Id_X)$  is just  $\widetilde{Cyl}(X)$ , the cubical cylinder object associated with  $X$ , that was introduced in 1.3.18.

**REMARK 1.7.7.** We will explain here why the cylinder considered in this section is called “cubical”.

Firstly, the category  $\square_1^\circ \mathcal{C}$  is a subcategory of the category of (all) “cubical diagrams” in  $\mathcal{C}$  introduced in [GN]. If  $\mathcal{C}$  has coproducts, there exists a functor

$$\square_1^\circ \mathcal{C} \rightarrow \Delta_e^\circ \mathcal{C}$$

that assigns to  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\mathcal{C}$  the strict simplicial object  $E(f, g)$  given by

$$Y \sqcup Z \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{f} \end{array} X \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} 0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} 0 \quad \dots \quad ,$$

where 0 is the initial object in  $\mathcal{C}$ .

On the other hand, we have the Dold-Puppe transform  $\pi : \Delta_e^\circ \mathcal{C} \rightarrow \Delta^\circ \mathcal{C}$  (see 1.1.16).

Setting  $\mathcal{C} = \Delta^\circ \mathcal{D}$ , then  $\pi(E(f, g)) \in \Delta^\circ \Delta^\circ \mathcal{D}$ , and its diagonal is just  $\widetilde{Cyl}(f, g)$ .

**PROPOSITION 1.7.8.** *The functor  $\widetilde{Cyl} : \square_1^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  commutes with coproducts, that is*

$$\widetilde{Cyl}(f, g) \sqcup \widetilde{Cyl}(f', g') \simeq \widetilde{Cyl}(f \sqcup f', g \sqcup g')$$

*Proof.* The statement is a consequence of the commutativity of  $\Phi$  and  $Tot$  with coproducts.  $\square$

**PROPOSITION 1.7.9.** *Given  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\square_1^\circ \Delta^\circ \mathcal{D}$ , the diagram (1.14) commutes up to simplicial homotopy equivalence, natural in  $(f, g)$ .*

*Proof.* Applying  $\widetilde{Cyl}$  to the following morphism of  $\square_1^\circ \Delta^\circ \mathcal{D}$

$$\begin{array}{ccccc} X & \xleftarrow{Id} & X & \xrightarrow{Id} & X \\ g \downarrow & & Id \downarrow & & f \downarrow \\ Z & \xleftarrow{g} & X & \xrightarrow{f} & Y, \end{array}$$

we obtain  $H : \widetilde{Cyl}(X) \rightarrow \widetilde{Cyl}(f, g)$ .

Following the notations introduced in 1.3.19, from the naturality of (1.14) we deduce that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{J} & \widetilde{Cyl}(X) & \xleftarrow{I} & X \\ g \downarrow & & H \downarrow & & f \downarrow \\ Z & \xrightarrow{j_Z} & \widetilde{Cyl}(f, g) & \xleftarrow{j_Y} & Y, \end{array}$$

commutes. Then  $H \circ I = j_Y \circ f$  and  $H \circ J = j_Z \circ g$ , therefore  $j_Y \circ f \sim j_Z \circ g$ .  $\square$

**REMARK 1.7.10.** The functor  $\widetilde{Cyl}$  also satisfies an analogue of 1.5.10, that will not be used in this work.

Given a commutative diagram in  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \rho' \\ Z & \xrightarrow{\rho} & T, \end{array} \tag{1.15}$$

there exists a morphism, natural in (1.15),  $H : \widetilde{Cyl}(f, g) \rightarrow T$  such that  $H \circ j_Z = \rho$  and  $H \circ j_Y = \rho'$ , that is

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow j_Y \\ Z & \xrightarrow{j_Z} & \widetilde{Cyl}(f, g) \end{array} \begin{array}{l} \nearrow \rho' \\ \searrow H \\ \nearrow \rho \end{array} \begin{array}{c} \\ \\ T \end{array}$$

Indeed, it is enough to consider  $H$  such that  $H_n|_{X_n} = \rho'_n f_n : X_n \rightarrow T_n$ , where  $X_n$  denotes a component of  $\widetilde{Cyl}(f, g)_n$ .

**REMARK 1.7.11.** A property of “factorization” (similar to 1.5.15) also holds for  $\widetilde{Cyl}$ , with respect to a diagram (1.11) of simplicial objects (introduced in 1.5.14). This property will not be used in this work.

Next we study the relationship between  $Cyl$  and  $\widetilde{Cyl}$  (in those cases in which they are comparable).

The following result is a direct consequence of the definitions of  $Cyl$  and  $\widetilde{Cyl}$ .

**PROPOSITION 1.7.12.** *If  $C \xleftarrow{g} A \xrightarrow{f} B$  is a diagram in  $\mathcal{D}$ , then*

$$Cyl(f \times \Delta, g \times \Delta) = \widetilde{Cyl}(f \times \Delta, g \times \Delta) .$$

**PROPOSITION 1.7.13.** *Let  $Z \xleftarrow{g} X \xrightarrow{f} Y$  be a diagram in  $\Delta^\circ \mathcal{D}$ . Then, following the notations given in 1.5.12, the diagonal simplicial object of the bisimplicial object  $Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times f, \Delta \times g)$  is equal to  $\widetilde{Cyl}(f, g)$ . Similarly,  $DCyl_{\Delta^\circ \mathcal{D}}^{(2)}(f \times \Delta, g \times \Delta) = \widetilde{Cyl}(f, g)$ .*

*Proof.* The bisimplicial object  $T = Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times f, \Delta \times g)$  is

$$T_{n,m} = Y_m \sqcup X_m^{(n-1)} \sqcup \dots \sqcup X_m^{(0)} \sqcup Z_m$$

where  $X_m^{(n-i)} = X_m \ \forall i = 1, \dots, n$ . The face maps  $d_i^{(2)} : T_{n,m} \rightarrow T_{n,m-1}$  respect to the second index are  $d_i^{(2)} = d_i^Y \sqcup d_i^X \sqcup \dots \sqcup d_i^X \sqcup d_i^Z$ , and analogously for the degeneracy maps  $s_k^{(2)} : T_{n,m} \rightarrow T_{n,m+1}$ .

On the other hand  $d_i^{(1)} : T_{n,m} \rightarrow T_{n-1,m}$  is  $d_i^{(1)}|_{Y_m} = Id$ ,  $d_i^{(1)}|_{Z_m} = Id$  and

$$d_i^{(1)}|_{X_m^{(n-k)}} = \begin{cases} Id : X_m^{(n-k)} \rightarrow X_m^{(n-k-1)} & \text{if } i \geq k \text{ and } (k, i) \neq (1, 0) \\ Id : X_m^{(n-k)} \rightarrow X_m^{(n-k)} & \text{if } i > k \text{ and } (k, i) \neq (n, n) \\ f_m : X_m^{(n-1)} \rightarrow Y_m & \text{if } (k, i) = (1, 0) \\ g_m : X_m^{(0)} \rightarrow Z_m & \text{if } (k, i) = (n, n) \end{cases} .$$

The degeneracy maps are built in a similar way using the definition of  $Cyl$ .

Clearly, the diagonal of  $T$ ,  $DT$ , coincides with  $\widetilde{Cyl}(f, g)$ . The last statement follows from the commutativity of diagram (1.10) and from the fact  $D\Gamma = D$ .  $\square$

**PROPOSITION 1.7.14.** *If  $X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  is a diagram in  $\Delta^\circ \mathcal{D}$ , then  $Cyl(f, \epsilon)$  is a retract of  $\widetilde{Cyl}(f, \epsilon)$ .*

*In other words, there exists morphisms  $\alpha : Cyl(f, \epsilon) \rightarrow \widetilde{Cyl}(f, \epsilon)$  and  $\beta : \widetilde{Cyl}(f, \epsilon) \rightarrow Cyl(f, \epsilon)$  such that  $\beta\alpha = Id$ .*

*In addition,  $\alpha$  and  $\beta$  are natural in  $(f, \epsilon)$  and commute with the inclusions of  $X_{-1}$  and  $Y$  into the respective cylinders.*



*Proof.* We have that

$$\alpha_n : Y_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1} \longrightarrow Y_n \sqcup X_n^{(n-1)} \sqcup \cdots \sqcup X_n^{(0)} \sqcup X_{-1}$$

is defined as the identity on  $Y_n$  and  $X_{-1}$ , and on  $X_{n-k}$  is  $(s_0)^k : X_{n-k} \rightarrow X_n^{(n-k)}$ . It holds that  $\alpha$  commutes with the face morphisms, since

$$\alpha_{n-1} d_i|_{X_{n-k}} = \begin{cases} (s_0)^k d_{i-k}^X & \text{if } i \geq k \\ (s_0)^{k-1} & \text{if } i < k \text{ and } (k, i) \neq (1, 0) \\ f_{n-1} & \text{if } (k, i) = (1, 0) \end{cases} ,$$

whereas

$$d_i \alpha_n|_{X_{n-k}} = \begin{cases} d_i^X (s_0)^k & \text{if } i \geq k \\ d_i (s_0)^k & \text{if } i < k \text{ and } (k, i) \neq (1, 0) \\ f_{n-1} d_0 s_0 & \text{if } (k, i) = (1, 0) \end{cases} .$$

The equality  $(s_0)^k d_{i-k} = d_i^X (s_0)^k$  follows from the iteration of the simplicial identity  $d_{j+1} s_0 = s_0 d_j$  if  $j > 1$ .

In addition, since  $(s_0)^l = s_{l-1}(s_0)^{l-1}$ , then  $d_i (s_0)^k = d_i (s_0)^i (s_0)^{k-i} = d_i s_{i-1} (s_0)^{k-1} = (s_0)^{k-1}$ . One can check similarly that  $\alpha$  commutes with the degeneracy maps.

On the other hand,  $\beta_n : Y_n \sqcup X_n^{(n-1)} \sqcup \cdots \sqcup X_n^{(0)} \sqcup X_{-1} \rightarrow Y_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_0 \sqcup X_{-1}$  is the identity on  $Y_n$  and  $X_{-1}$ , and on  $X_n^{(n-k)}$  is  $(d_0)^k : X_n^{(n-k)} \rightarrow X_{n-k}$ .

We deduce again from the simplicial identities that  $\beta$  is in fact a morphism in  $\Delta^\circ \mathcal{D}$ , as well as the equality  $\beta \alpha = Id$  holds, and it is clear that the inclusions of  $Y$  and  $X_{-1}$  commutes with both morphisms.  $\square$

**COROLLARY 1.7.15.** *The diagram (1.8) given in 1.5.6 commutes up to simplicial homotopy.*

*Proof.* Let  $X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  be a diagram in  $\Delta^\circ \mathcal{D}$ . It follows from 1.7.9 that there exists  $R : \widetilde{Cyl}(X) \rightarrow \widetilde{Cyl}(f, \epsilon)$  such that  $R \circ I = j_Y \circ f$  and  $R \circ J = j_{X_{-1}} \circ \epsilon$ .

Therefore  $H = \beta \circ R : \widetilde{Cyl}(X) \rightarrow Cyl(f, \epsilon)$  is such that  $H \circ I = i_Y \circ f$  and  $H \circ J = i_{X_{-1}} \circ \epsilon$ .  $\square$

## Chapter 2

# Simplicial Descent Categories

The notion of (co)simplicial descent category is widely based in the one of “(co)homological descent category”, introduced in [GN]. In loc. cit. the basic objects are diagrams of “cubical” nature instead of simplicial objects in a fixed category.

### 2.1 Definition

**DEFINITION 2.1.1.** Consider a category  $\mathcal{D}$  and a class of morphisms  $E$  in  $\mathcal{D}$ . We will denote by  $Ho\mathcal{D}$  the localization of  $\mathcal{D}$  with respect to  $E$ , and by  $\gamma : \mathcal{D} \rightarrow Ho\mathcal{D}$  the canonical functor. The class  $E$  is saturated if

$$\text{a morphism } f \text{ is in } E \iff \gamma(f) \text{ is an isomorphism in } Ho\mathcal{D} .$$

Equivalently,  $E$  is saturated if  $E = \gamma^{-1}\{ \text{isomorphisms of } Ho\mathcal{D} \}$ .

**REMARK 2.1.2.**

i) If  $E$  is saturated, the “2 out of 3” property holds for  $E$ . That is, if two of the morphisms  $f$ ,  $g$  or  $gf$  are in  $E$  then so is the third.

ii) An enough (and necessary) condition for  $E$  being saturated is that the morphisms in  $E$  are just those that are mapped by a certain functor into isomorphisms. In other words:

If  $F : \mathcal{D} \rightarrow \mathcal{C}$  is a functor and  $E = \{f \mid F(f) \text{ is an isomorphism} \}$  then  $E$  is saturated.

The following lemma will be needed later, whose (trivial) proof is left to the reader.

**LEMMA 2.1.3.** *If  $E$  is a saturated class of morphisms in a category  $\mathcal{D}$  with final object 1, then the class of acyclic objects of  $\mathcal{D}$  (with respect to  $E$ )*

$$\mathcal{A} = \{\text{objects } A \text{ of } \mathcal{D} \mid A \rightarrow 1 \text{ is an equivalence}\}$$

*is closed under retracts.*

*More specifically, if  $A \xrightarrow{r} B \xrightarrow{p} A$  is such that  $p \circ r = Id_A$  and  $B \rightarrow 1$  is an equivalence, then  $A \rightarrow 1$  is also an equivalence.*

*Proof.* By the 2 out of 3 property, it is enough to see that  $r$  is an equivalence. Let  $\xi : B \rightarrow 1$  be the trivial morphism. Then  $\xi \circ r \circ p = \xi : B \rightarrow 1$ , since 1 is final object. Therefore  $\rho = r \circ p : B \rightarrow B$  is in  $E$ .

Thus, the equalities  $p \circ r = Id_A$  and  $r \circ (p \circ \rho^{-1}) = Id_B$  hold in  $Ho\mathcal{D}$ . So  $p$  is a right inverse of  $r$  and  $p\rho^{-1}$  is a left inverse of  $r$ . Consequently  $r$  is an isomorphism in  $Ho\mathcal{D}$  with  $p\rho^{-1} = p$  as inverse. Since  $E$  is saturated, we deduce that  $r \in E$ .  $\square$

Before going into details with the notion of simplicial descent categories, we introduce the following notations.

**(2.1.4)** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories with finite coproducts (in particular initial object 0).

Note that every functor  $\psi : \mathcal{C} \rightarrow \mathcal{D}$  is (lax) monoidal with respect to the coproduct. The Künneth morphism is the one given by the universal property of the coproduct

$$\sigma_{X,Y} : \psi(X) \sqcup \psi(Y) \rightarrow \psi(X \sqcup Y) \quad \forall X, Y \in \mathcal{C} \quad ; \quad \sigma_0 := 0 \rightarrow \psi(0) .$$

The morphism  $\sigma_{X,Y}$  is the unique morphism such that the diagram

$$\begin{array}{ccc} \psi(X) & \xrightarrow{\psi(i_X)} & \\ & \searrow i_{\psi(X)} & \downarrow \sigma_{X,Y} \\ & \psi(X) \sqcup \psi(Y) & \rightarrow \psi(X \sqcup Y) \\ & \nearrow i_{\psi(Y)} & \uparrow \\ \psi(Y) & \xrightarrow{\psi(i_Y)} & \end{array}$$

commutes. We will denote this natural transformation by  $\sigma_\psi$ , or just  $\sigma$  if  $\psi$  is understood.

If  $E$  is a class of morphisms of  $\mathcal{D}$ , the functor  $\psi$  is said to be *quasi-strict monoidal* (with respect to  $E$ ) if  $\sigma_{X,Y}$  and  $\sigma_0$  belongs to  $E$  for every objects  $X$  and  $Y$  in  $\mathcal{C}$ .

Dually, if  $\mathcal{C}$  and  $\mathcal{D}$  have finite products then every functor  $\psi : \mathcal{C} \rightarrow \mathcal{D}$  is (lax) comonoidal with respect to the product. This time the Künneth morphism is

the canonical morphism  $\sigma_\psi : \psi(X \times Y) \rightarrow \psi(X) \times \psi(Y)$  given by the universal property of the product.

**(2.1.5)** Assume that a functor  $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  is given. Under the notations introduced in 1.1.21, the image under

$$\Delta^\circ \mathbf{s} : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$$

of a bisimplicial object  $T$  in  $\mathcal{D}$  is the simplicial object

$$(\Delta^\circ \mathbf{s}(T))_n = \mathbf{s}(T_{n,\cdot}) = \mathbf{s}(m \rightarrow T_{n,m}) .$$

**DEFINITION 2.1.6** (Simplicial descent category).

A (*simplicial*) *descent category* consists of the data  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  where:

**(SDC 1)**  $\mathcal{D}$  is a category with finite coproducts (in particular with initial object 0) and with final object 1.

**(SDC 2)**  $\mathbf{E}$  is a saturated class of morphisms in  $\mathcal{D}$ , stable by coproducts (that is  $\mathbf{E} \sqcup \mathbf{E} \subseteq \mathbf{E}$ ). The morphisms in  $\mathbf{E}$  will be called *equivalences*.

**(SDC 3) Additivity:** The *simple* functor  $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  commutes with coproducts up to equivalence. In other words, the canonical morphism  $\mathbf{s}X \sqcup \mathbf{s}Y \rightarrow \mathbf{s}(X \sqcup Y)$  is in  $\mathbf{E}$  for all  $X, Y$  in  $\Delta^\circ \mathcal{D}$ .

**(SDC 4) Factorization:** Let  $\mathbf{D} : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  be the diagonal functor. Consider the functors  $\mathbf{s}(\Delta^\circ \mathbf{s}), \mathbf{sD} : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ . Then  $\mu$  is a natural transformation  $\mu : \mathbf{sD} \rightarrow \mathbf{s}(\Delta^\circ \mathbf{s})$  such that  $\mu_T \in \mathbf{E}$  for all  $T \in \Delta^\circ \Delta^\circ \mathcal{D}$ .

**(SDC 5) Normalization:**  $\lambda : \mathbf{s}(- \times \Delta) \rightarrow Id_{\mathcal{D}}$  is a natural transformation compatible with  $\mu$  and such that  $\lambda_X \in \mathbf{E}$  for all  $X \in \mathcal{D}$ .

**(SDC 6) Exactness:** If  $f : X \rightarrow Y$  is a morphism in  $\Delta^\circ \mathcal{D}$  with  $f_n \in \mathbf{E} \forall n$  then  $\mathbf{s}(f) \in \mathbf{E}$ .

**(SDC 7) Acyclicity:** If  $f : X \rightarrow Y$  is a morphism in  $\Delta^\circ \mathcal{D}$ , then  $\mathbf{s}f \in \mathbf{E}$  if and only if the simple of its simplicial cone is acyclic, that means that  $\mathbf{s}(Cf) \rightarrow 1$  is an equivalence.

**(SDC 8) Symmetry:**  $\mathbf{s}\Upsilon f \in \mathbf{E}$  if (and only if)  $\mathbf{s}f \in \mathbf{E}$ , where  $\Upsilon$  is the functor  $\Upsilon : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  that reverses the order of the face and degeneracy maps in a simplicial object in  $\mathcal{D}$ .

(2.1.7)[Compatibility between  $\lambda$  and  $\mu$ ] Given  $X \in \Delta^\circ \mathcal{D}$ , we have that

$$\begin{aligned} \mathbf{s} \circ \Delta^\circ \mathbf{s}(X \times \Delta) &= \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow X_n)) = \mathbf{s}(n \rightarrow \mathbf{s}(X_n \times \Delta)) \\ \mathbf{s} \circ \Delta^\circ \mathbf{s}(\Delta \times X) &= \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow X_m)) = \mathbf{s}(\mathbf{s}(X) \times \Delta) . \end{aligned}$$

In the first case, the morphisms  $\lambda_{X_n} : \mathbf{s}(X_n \times \Delta) \rightarrow X_n$  give rise to a morphism of simplicial objects.

The compatibility condition between  $\lambda$  and  $\mu$  means that the following compositions must be equal to the identity in  $\mathcal{D}$ :

$$\begin{aligned} \mathbf{s}X &\xrightarrow{\mu_{\Delta \times X}} \mathbf{s}((\mathbf{s}X) \times \Delta) \xrightarrow{\lambda_{\mathbf{s}X}} \mathbf{s}X \\ \mathbf{s}X &\xrightarrow{\mu_{X \times \Delta}} \mathbf{s}(n \rightarrow \mathbf{s}(X_n \times \Delta)) \xrightarrow{\mathbf{s}(\lambda_{X_n})} \mathbf{s}X . \end{aligned} \tag{2.1}$$

**REMARK 2.1.8** (Comments on the symmetry axiom).

We will use later the following property of the image under the simple functor of the simplicial cylinder associated with a morphism  $f : X \rightarrow Y$  and with an augmentation  $\epsilon : X \rightarrow X_{-1} \times \Delta$ :

(\*) The simple of  $f$  is an equivalence when the simple of the canonical inclusion  $i_{X_{-1}} : X_{-1} \times \Delta \rightarrow \text{Cyl}(f, \epsilon)$  is so.

The converse property will be also needed to prove the “transfer lemma” 2.5.8, at least under some extra hypothesis.

The symmetry axiom is imposed in order to have the converse statement of (\*) (see section 2.4).

However, other possibility is to impose the axiom:  $sf \in \mathbf{E}$  if and only if  $\mathbf{s}(i_{X_{-1}}) \in \mathbf{E}$ , and remove the symmetry axiom from the notion of simplicial descent category (in this case (SDC 7) holds setting  $X_{-1} = 1$ ).

We decide to do it in this way because the aim of this work is just to establish a set of axioms ensuring the desired properties, and we would like these axioms to be “less restrictive as possible”.

An alternative to (SDC 8) is the existence of an isomorphism of functors between  $\mathbf{s}$  and  $\mathbf{s}\Upsilon : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ . But even this property holds in many of our examples, it is not true for  $\mathcal{D} = \text{Set}$  and the diagonal  $D : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  as simple functor.

**REMARK 2.1.9.** We consider in  $\mathcal{D}$ ,  $\Delta^\circ \mathcal{D}$  and  $\Delta^\circ \Delta^\circ \mathcal{D}$  the trivial monoidal structures coming from the coproduct. Then we have automatically that  $\mathbf{s}$  and  $\Delta^\circ \mathbf{s}$  are (lax) monoidal functors, with  $\sigma = \sigma_{\mathbf{s}}$  and  $\sigma_{\Delta^\circ \mathbf{s}}$  as respective Künneth

morphisms (see 2.1.4).

In addition the natural transformations  $\lambda$  and  $\mu$  are also monoidal.

That is to say, these transformations are compatible with  $\sigma$  in the following sense. Given objects  $X, Y$  in  $\mathcal{D}$ , the diagram

$$\begin{array}{ccc} \mathbf{s}(X \times \Delta) \sqcup \mathbf{s}(Y \times \Delta) & \xrightarrow{\sigma} & \mathbf{s}((X \sqcup Y) \times \Delta) \\ & \searrow \lambda_X \sqcup \lambda_Y & \downarrow \lambda_{X \sqcup Y} \\ & & X \sqcup Y \end{array} \quad (2.2)$$

commutes. On the other hand, let  $Z$  and  $T$  be bisimplicial objects in  $\mathcal{D}$ . Then we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{s}DZ \sqcup \mathbf{s}DT & \xrightarrow{\sigma} & \mathbf{s}D(Z \sqcup T) \\ \mu_Z \sqcup \mu_T \downarrow & & \mu_{Z \sqcup T} \downarrow \\ \mathbf{s}(\Delta^\circ \mathbf{s}Z) \sqcup \mathbf{s}(\Delta^\circ \mathbf{s}T) & \xrightarrow{\sigma \Delta^\circ \sigma} & \mathbf{s}(\Delta^\circ \mathbf{s}(Z \sqcup T)). \end{array}$$

**REMARK 2.1.10. Factorization:**

Recall the functor  $\Gamma : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}$  that swaps the indexes in a bisimplicial object in  $\mathcal{D}$ . In the factorization axiom we may also consider  $\mathbf{s}(\Delta^\circ \mathbf{s})\Gamma : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ .

Assuming (SDC 4), since  $D\Gamma = D$  we deduce the existence of the natural transformation  $\mu' : \mathbf{s}D \rightarrow \mathbf{s}(\Delta^\circ \mathbf{s})\Gamma$  given by  $\mu'(Z) = \mu(\Gamma Z)$  and such that  $\mu'(Z) \in E$ ,  $\forall Z \in \Delta^\circ \Delta^\circ \mathcal{D}$ . Then

$$\mathbf{s}(\Delta^\circ \mathbf{s})Z \xleftarrow{\mu(Z)} DZ \xrightarrow{\mu'(Z)} \mathbf{s}(\Delta^\circ \mathbf{s})(\Gamma Z) \quad .$$

**PROPOSITION 2.1.11.** *The axiom (SDC 6) in the notion of simplicial descent category can be replaced by the following alternative axiom*

**(SDC 6)'** *If  $X$  is an object in  $\Delta^\circ \mathcal{D}$  such that  $X_n \rightarrow 1$  is an equivalence for every  $n$ , then  $\mathbf{s}X \rightarrow 1$  is also in  $E$ .*

*Proof.* Assume that  $\mathcal{D}$  satisfies (SDC 6)' instead of (SDC 6), together with the remaining axioms of simplicial descent category.

Let  $f : X \rightarrow Y$  be a morphism in  $\Delta^\circ \mathcal{D}$  with  $f_n \in E$  for all  $n$ .

Then, for a fixed  $n \geq 0$  we have that  $\mathbf{s}(f_n \times \Delta) : \mathbf{s}(X \times \Delta) \rightarrow \mathbf{s}(Y \times \Delta)$  is an equivalence, since  $\lambda_Y \circ \mathbf{s}(f_n \times \Delta) = f_n \circ \lambda_X$  and the 2 out of 3 property holds for  $E$ .

Hence, it follows from the acyclicity axiom that  $\mathbf{s}C(f_n \times \Delta) \rightarrow 1$  is in  $E$ , for

every  $n$ .

Consider now the bisimplicial object  $T \in \Delta^\circ \Delta^\circ \mathcal{D}$  defined by

$$T_{n,m} = C(f_n \times \Delta)_m = Y_n \sqcup \coprod_{i=1}^m X_n \sqcup 1 .$$

Equivalently,  $T$  is the image under  $Cyl^{(2)}$  of  $1 \times \Delta \times \Delta \longleftarrow X \times \Delta \xrightarrow{f \times \Delta} Y \times \Delta$ , (see 1.5.12). Therefore

$$\mathbf{s}(\Delta^\circ \mathbf{s}(T)) = \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow C(f_n \times \Delta)_m)) = \mathbf{s}(n \rightarrow \mathbf{s}(C(f_n \times \Delta))) ,$$

and from (SDC 6)' we deduce that  $\mathbf{s}\Delta^\circ \mathbf{s}(T) \rightarrow 1$  is an equivalence.

Moreover, by the factorization axiom we have that  $\mu_T : \mathbf{s}DT \rightarrow \mathbf{s}(\Delta^\circ \mathbf{s}(T))$  is in  $\mathbf{E}$ , and again the 2 out of 3 property implies that the trivial morphism  $\rho : \mathbf{s}DT \rightarrow 1$  is also in  $\mathbf{E}$ .

On the other hand,  $DT = DCyl^{(2)}(1 \times \Delta \times \Delta \longleftarrow X \times \Delta \xrightarrow{f \times \Delta} Y \times \Delta)$  that agrees with  $\widetilde{C}f$ , the image under  $\widetilde{Cyl}$  of  $1 \times \Delta \longleftarrow X \xrightarrow{f} Y$ , because of proposition 1.7.13.

From 1.7.14 it follows that  $\mathbf{s}Cf$  is a retract of  $\mathbf{s}\widetilde{C}f = \mathbf{s}DT$ , that is an acyclic object. Then we conclude by lemma 2.1.3 that  $\mathbf{s}Cf$  is acyclic, and using the acyclicity axiom we get that  $\mathbf{s}f \in \mathbf{E}$ .  $\square$

The following properties are direct consequences of the axioms.

**PROPOSITION 2.1.12.** *If  $f : X \rightarrow Y$  is a morphism between simplicial objects in a simplicial descent category  $\mathcal{D}$ , then*

$$\mathbf{s}f \in \mathbf{E} \text{ if and only if } \mathbf{s}(C'f) \rightarrow 1 \text{ is an equivalence,}$$

where  $C'$  is the “symmetric” notion of simplicial cone, given in 1.6.6.

*Proof.* It follows from the symmetry axiom that  $\mathbf{s}f \in \mathbf{E}$  if and only if  $\mathbf{s}\Upsilon f \in \mathbf{E}$ . By the acyclicity axiom, this happens if and only if  $\mathbf{s}C(\Upsilon f) \rightarrow 1$  is an equivalence. If  $\tau : C(\Upsilon f) \rightarrow 1 \times \Delta$  is the trivial morphism, then  $\mathbf{s}C(\Upsilon f) \rightarrow 1$  is an equivalence if and only if  $\mathbf{s}(\tau) : \mathbf{s}C(\Upsilon f) \rightarrow \mathbf{s}(1 \times \Delta)$  is so, because the morphism  $\mathbf{s}(1 \times \Delta) \rightarrow 1$  is in  $\mathbf{E}$  by the normalization axiom.

Again this condition is equivalent to the fact that  $\mathbf{s}(\Upsilon\tau) : \mathbf{s}(\Upsilon C(\Upsilon f)) \rightarrow \mathbf{s}(1 \times \Delta)$  is an equivalence, since  $\Upsilon(1 \times \Delta) = 1 \times \Delta$ . Finally, by definition  $C'f = \Upsilon C\Upsilon f$ , and the statement follows from the acyclicity of the object  $\mathbf{s}(1 \times \Delta)$ .  $\square$

**PROPOSITION 2.1.13.** *Let  $\mathcal{D}$  be a simplicial descent category. Consider a morphism  $(\alpha, \beta, \gamma) : D \rightarrow D'$  in  $\Omega(\mathcal{D})$ ,*

$$\begin{array}{ccccc} X_{-1} \times \Delta & \xleftarrow{\epsilon} & X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ X'_{-1} \times \Delta & \xleftarrow{\epsilon'} & X' & \xrightarrow{f'} & Y', \end{array}$$

*such that  $\alpha_n, \beta_n$  and  $\gamma_n$  are in  $E$  for all  $n$ . Then the induced morphism  $s(Cyl(\alpha, \beta, \gamma)) : s(Cyl(D)) \rightarrow s(Cyl(D'))$  is also in  $E$ .*

*Proof.* By definition  $Cyl(\alpha, \beta, \gamma)_n = \gamma_n \sqcup \beta_{n-1} \sqcup \cdots \sqcup \beta_0 \sqcup \alpha$ . So it follows from (SDC 2) that  $Cyl(\alpha, \beta, \gamma)_n \in E \forall n$ , and from (SDC 6) that  $s(Cyl(\alpha, \beta, \gamma)) \in E$ .  $\square$

**COROLLARY 2.1.14.** *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \beta \downarrow & & \downarrow \gamma \\ X' & \xrightarrow{f'} & Y', \end{array}$$

*is a morphism in  $Fl(\Delta^\circ \mathcal{D})$  such that  $\beta_n$  and  $\gamma_n$  are equivalences for all  $n$ , then  $sC(\beta, \gamma) : s(Cf) \rightarrow s(Cf')$  is also in  $E$ .*

*Proof.* Just set  $X_{-1} = 1$  and  $\alpha = Id$  in the last proposition.  $\square$

**PROPOSITION 2.1.15.** *If  $I$  is a small category and  $(\mathcal{D}, E_{\mathcal{D}}, s_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}})$  is a simplicial descent category then the category of functors from  $I$  to  $\mathcal{D}$ ,  $ID$ , has a natural structure of simplicial descent category.*

*Given  $X : \Delta^\circ \rightarrow ID$ , the image under the simple functor in  $ID$ ,  $s_{ID}$ , of a simplicial object  $X$  in  $ID$  is defined as*

$$(s_{ID}(X))(i) = s_{\mathcal{D}}(n \rightarrow X_n(i))$$

*and  $E_{ID} = \{f \text{ such that } f(i) \in E_{\mathcal{D}} \forall i \in I\}$ .*

*Proof.* Define also  $\lambda_{ID}$  and  $\rho_{ID}$  through the identification

$$\Delta^\circ(ID) \equiv I\Delta^\circ \mathcal{D} . \quad (2.3)$$

Then, (SDC 1) is clear, since the coproduct in  $ID$  is defined degreewise. The verification of the axioms (SDC 3),  $\dots$ , (SDC 6) and (SDC 8) is straightforward. To see (SDC 2), let us check that  $E_{ID}$  is a saturated class.



Let  $\gamma_{I\mathcal{D}} : I\mathcal{D} \rightarrow I\mathcal{D}[\mathbf{E}_{I\mathcal{D}}^{-1}]$  and  $\gamma_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}[\mathbf{E}_{\mathcal{D}}^{-1}]$  be the localizations of  $I\mathcal{D}$  and  $\mathcal{D}$  with respect to  $\mathbf{E}_{I\mathcal{D}}$  and  $\mathbf{E}_{\mathcal{D}}$  respectively.

Given an object  $j$  in  $I$ , consider the “evaluation” functor  $\pi_j : I\mathcal{D} \rightarrow \mathcal{D}$ , given by  $\pi_j(P) = P(j)$ .

Then  $\gamma_{\mathcal{D}}\pi_j(\mathbf{E}_{I\mathcal{D}}) \subseteq \{\text{isomorphisms of } \mathcal{D}[\mathbf{E}_{\mathcal{D}}^{-1}]\}$ , and the composition  $\gamma_{\mathcal{D}}\pi_j$  gives rise to the following commutative diagram of functors,

$$\begin{array}{ccc} I\mathcal{D} & \xrightarrow{\pi_j} & \mathcal{D} \\ \gamma_{I\mathcal{D}} \downarrow & & \downarrow \gamma_{\mathcal{D}} \\ I\mathcal{D}[\mathbf{E}_{I\mathcal{D}}^{-1}] & \xrightarrow{\bar{\pi}_j} & \mathcal{D}[\mathbf{E}_{\mathcal{D}}^{-1}] . \end{array}$$

Therefore, if  $f$  is a morphism in  $I\mathcal{D}$  such that  $\gamma_{I\mathcal{D}}(f)$  is an isomorphism, it follows that  $\gamma_{\mathcal{D}}(f(j))$  is so for every  $j \in I$ .

Hence,  $f(j) \in \mathbf{E}_{\mathcal{D}} \forall j$  and by definition  $f \in \mathbf{E}_{I\mathcal{D}}$ .

To check (SDC 7) it is enough to note that, if we denote by  $C_{I\mathcal{D}} : Fl(\Delta^\circ I\mathcal{D}) \rightarrow \Delta^\circ I\mathcal{D}$  and  $C_{\mathcal{D}} : Fl(\Delta^\circ \mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  the respective cone functors, then (by definition of the coproduct in  $I\mathcal{D}$ ), it holds that  $[C_{I\mathcal{D}}(f)](j) = C_{\mathcal{D}}(f(j))$ .

Finally, (SDC 8) follows from the equality  $(\mathbf{s}_{I\mathcal{D}}(\Upsilon f))(i) = \mathbf{s}_{\mathcal{D}}(\Upsilon f(i))$ , for each  $i \in I$ .  $\square$

**COROLLARY 2.1.16.** *If  $\mathcal{D}$  is a (simplicial) descent category then  $\Delta^\circ \mathcal{D}$  is so, where the simple functor  $\tilde{\mathbf{s}}$  is defined as*

$$[\tilde{\mathbf{s}}(Z)]_n = \mathbf{s}(m \rightarrow Z_{m,n}) \text{ for all } Z \in \Delta^\circ \Delta^\circ \mathcal{D}$$

and where the class of equivalences is

$$\mathbf{E}_{\Delta^\circ \mathcal{D}} = \{f \text{ such that } f_n \in \mathbf{E}_{\mathcal{D}} \forall n\} .$$

**REMARK 2.1.17.** Following the notations in (2.1.5) and 2.1.10, the functor  $\tilde{\mathbf{s}} : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  is the composition  $\Delta^\circ \mathbf{s}_\circ \Gamma : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$ . This follows from the identification (2.3), that now is just  $\Gamma$ .

A natural question is if it is also possible to consider  $\Delta^\circ \mathbf{s}$  as a simple. However, this time the transformation  $\lambda$  should be a morphism in  $\Delta^\circ \mathcal{D}$  relating  $(\mathbf{s}X) \times \Delta$  to  $X$ , and in general this transformation  $\lambda$  does not exist.

## Cosimplicial Descent Categories

**DEFINITION 2.1.18.** A cosimplicial descent category consists of the data  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  where  $\mathcal{D}$  is a category,  $\mathbf{s} : \Delta\mathcal{D} \rightarrow \mathcal{D}$  is a functor,  $\mathbf{E}$  is a class of morphisms in

$\mathcal{D}$ , and  $\mu : \mathbf{s}\Delta^\circ \mathbf{s} \rightarrow \mathbf{s}\mathcal{D}$  and  $\lambda : Id_{\mathcal{D}} \rightarrow \mathbf{s}(- \times \Delta)$  are natural transformations, such that  $\mathcal{D}^\circ$ , the opposite category of  $\mathcal{D}$ , together with  $(\mathbf{E}^\circ, \mathbf{s}^\circ, \mu^\circ, \lambda^\circ)$ , induced by  $(\mathbf{E}, \mathbf{s}, \mu, \lambda)$  in  $\mathcal{D}^\circ$ , is a simplicial descent category.

More specifically, a cosimplicial descent category is the data  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  where:

- (CDC 1)  $\mathcal{D}$  is a category with finite products and initial object 0.
- (CDC 2)  $\mathbf{E}$  is a saturated class of morphisms in  $\mathcal{D}$ , stable by products. That is, given  $f, g \in \mathbf{E}$  then  $f \sqcap g \in \mathbf{E}$ .
- (CDC 3) **Additivity:**  $\mathbf{s} : \Delta\mathcal{D} \rightarrow \mathcal{D}$  is a quasi-strict comonoidal functor with respect to the product.
- (CDC 4) **Factorization:** If  $D : \Delta\Delta\mathcal{D} \rightarrow \Delta\mathcal{D}$  is the diagonal functor, consider  $\mathbf{s}(\Delta\mathbf{s}), \mathbf{s}D : \Delta\Delta\mathcal{D} \rightarrow \mathcal{D}$ . Then  $\mu : \mathbf{s}(\Delta\mathbf{s}) \rightarrow \mathbf{s}D$  is a natural transformation such that  $\mu(Z) \in \mathbf{E}$  for every  $Z \in \Delta\Delta\mathcal{D}$ .
- (CDC 5) **Normalization:**  $\lambda : Id_{\mathcal{D}} \rightarrow \mathbf{s}(- \times \Delta)$  is a natural transformation, compatible with  $\mu$ , such that for every  $X \in \mathcal{D}$ ,  $\lambda(X) \in \mathbf{E}$ .
- (CDC 6) **Exactness:** Given  $f : X \rightarrow Y$  in  $\Delta\mathcal{D}$  such that  $f_n \in \mathbf{E} \forall n$  then  $\mathbf{s}(f) \in \mathbf{E}$ .
- (CDC 7) **Acyclicity:** If  $f : X \rightarrow Y$  is a morphism in  $\Delta\mathcal{D}$ , then  $\mathbf{s}f \in \mathbf{E}$  if and only if the simple of its cosimplicial path object is acyclic. That is, if and only if  $0 \rightarrow \mathbf{s}(Path f)$  is an equivalence.
- (CDC 8) **Symmetry:** it holds that  $\mathbf{s}\Upsilon f \in \mathbf{E}$  if (and only if)  $\mathbf{s}f \in \mathbf{E}$ .

## 2.2 Cone and Cylinder objects in a simplicial descent category

From now on,  $\mathcal{D}$  will denote a simplicial descent category. The simplicial cone and cylinder functors in the category  $\Delta^\circ\mathcal{D}$  (section 1.5) induce cone and cylinder functors in  $\mathcal{D}$  through the constant and simple functors.

In addition, since  $\mathcal{D}$  is a simplicial descent category, these functors satisfy the “usual” properties (as in the chain complex case or topological case).

Of course, dual properties to those contained in this section remain valid in the cosimplicial setting.

Again, we mean by  $Ho\mathcal{D}$  the localized category of  $\mathcal{D}$  with respect to  $\mathbf{E}$ .

**DEFINITION 2.2.1.** Let  $R : \mathcal{D} \rightarrow \mathcal{D}$  be the functor defined as  $R = \mathbf{s}_\circ(- \times \Delta)$ . The normalization axiom provides the natural transformation  $\lambda : R \rightarrow Id_{\mathcal{D}}$ , such that  $\lambda_X : RX \rightarrow X \in \mathbf{E} \forall X$  in  $\mathcal{D}$ .

Therefore, given a morphism  $f$  in  $\mathcal{D}$  it follows from the 2 out of 3 property and from the naturality of  $\lambda$  that  $f \in E$  if and only if  $Rf \in E$ .

**(2.2.2)** Following the notations introduced in 1.5.1 and 1.7.1, the functor  $- \times \Delta : \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  induces  $- \times \Delta : \square_1^\circ \mathcal{D} \rightarrow \Omega(\mathcal{D})$ , as well as  $s : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  induces  $s : Co\Omega(\mathcal{D}) \rightarrow \square_1 \mathcal{D}$ .

Again, we identify  $Fl(\mathcal{D})$  with the full subcategory of  $\square_1^\circ \mathcal{D}$  whose objects are the diagrams  $1 \leftarrow X \rightarrow Y$ .

**DEFINITION 2.2.3** (cone and cylinder functors).

We define the cylinder functor  $cyl : \square_1^\circ \mathcal{D} \rightarrow \mathcal{D}$  as the composition

$$\square_1^\circ \mathcal{D} \xrightarrow{- \times \Delta} \Omega(\mathcal{D}) \xrightarrow{Cyl} \Delta^\circ \mathcal{D} \xrightarrow{s} \mathcal{D} .$$

More specifically, given morphisms  $f : A \rightarrow B$  and  $g : A \rightarrow C$  in  $\mathcal{D}$ , the cylinder of  $(f, g) \in \square_1^\circ \mathcal{D}$  is the image under the simple functor of the simplicial object  $Cyl(f \times \Delta, g \times \Delta)$ .

If  $1$  is a final object in  $\mathcal{D}$ , the cone functor  $c : Fl(\mathcal{D}) \rightarrow \mathcal{D}$  is defined in a similar way as  $c = s \circ C \circ (- \times \Delta)$ . Hence, the cone of  $f : A \rightarrow B$  is the simple of  $C(f \times \Delta) = Cyl(f \times \Delta, A \times \Delta \rightarrow 1 \times \Delta)$ .

**(2.2.4)** We deduce from 1.5.6 that if  $X \xleftarrow{g} Y \xrightarrow{f} Z$  is in  $\square_1^\circ \mathcal{D}$ , then applying  $cyl$  we obtain an object in  $\square_1 \mathcal{D}$ , natural in  $(f, g)$ , consisting of

$$RX \xrightarrow{I_X} cyl(f, g) \xleftarrow{I_Z} RZ .$$

In the first chapter we have developed other notions of simplicial cylinder different from  $Cyl$ , that are  $\widetilde{Cyl}$  and  $Cyl'$  (given respectively in 1.7.3 and 1.6.2). However, the definition of  $cyl$  “does not depend” on the choice between  $Cyl$  and  $\widetilde{Cyl}$ . On the other hand, if we take  $Cyl'$  instead of  $Cyl$  in the definition of  $cyl$  (that is, its conjugate with respect to  $\Upsilon$ ), the result is the same as if we swap the variables  $f$  and  $g$  in  $cyl$ .

**PROPOSITION 2.2.5.**

a) Given an object  $C \xleftarrow{g} A \xrightarrow{f} B$  in  $\square_1^\circ \mathcal{D}$ , it holds that

$$cyl(f, g) = s(\widetilde{Cyl}(f \times \Delta, g \times \Delta)) .$$

b) There exists a natural isomorphism in  $(f, g)$

$$\text{cyl}(f, g) \simeq \text{cyl}'(g, f) ,$$

where  $\text{cyl}'(g, f) = \mathbf{s}(\text{Cyl}'(g \times \Delta, f \times \Delta))$ . Moreover this isomorphism commutes with the canonical inclusions of RB and RC into the respective cylinders.

*Proof.* Part a) is a direct consequence of 1.7.12, whereas b) follows from 1.6.9.  $\square$

Next we introduce some properties of the functors  $\text{cyl}$  and  $c$  that can be deduced trivially from the definitions.

**PROPOSITION 2.2.6.** *Consider a morphism in  $\square_1^\circ \mathcal{D}$*

$$\begin{array}{ccccc} X & \xleftarrow{g} & Y & \xrightarrow{f} & Z \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ X' & \xleftarrow{g'} & Y' & \xrightarrow{f'} & Z' , \end{array}$$

such that  $\alpha, \beta, \gamma$  are equivalences. Then the induced morphism  $\text{cyl}(f, g) \rightarrow \text{cyl}(f', g')$  is also an equivalence.

*Proof.* The statement follows trivially from the definition of  $\text{cyl}$  and from 2.1.13.  $\square$

**COROLLARY 2.2.7.** *Consider a commutative square in  $\mathcal{D}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \beta \downarrow \\ X' & \xrightarrow{f'} & Y' , \end{array}$$

such that  $\alpha$  and  $\beta$  are equivalences. Then the induced morphism  $c(f) \rightarrow c(f')$  is also an equivalence.

**PROPOSITION 2.2.8** (Additivity of the cone and cylinder functors).

A) The cylinder functor is “additive up to equivalence”. In other words, given  $(f, g), (f', g') \in \square_1^\circ \mathcal{D}$ , then the natural morphism

$$\sigma_{\text{cyl}} : \text{cyl}(f, g) \sqcup \text{cyl}(f', g') \rightarrow \text{cyl}(f \sqcup f', g \sqcup g') ,$$

defined as in 2.1.4, is an equivalence.

B) The morphism

$$\sigma_c : c(f) \sqcup c(f') \rightarrow c(f \sqcup f')$$

is an equivalence for any  $f, f' \in \text{Fl}(\mathcal{D})$  if and only if  $1 \sqcup 1 \rightarrow 1$  is an equivalence.

*Proof.* The first statement follows from the additivity of the simplicial cylinder functor  $Cyl$  together with the axiom (SDC 3).

Indeed, given  $(f, g), (f', g') \in \square_1^\circ \mathcal{D}$ , we deduce from proposition 1.5.8 that

$$\sigma_{Cyl} : Cyl(f \times \Delta, g \times \Delta) \sqcup Cyl(f' \times \Delta, g' \times \Delta) \rightarrow Cyl((f \sqcup f') \times \Delta, (g \sqcup g') \times \Delta)$$

is an isomorphism. Therefore the following morphism is also an isomorphism

$$\mathbf{s}\sigma_{Cyl} : \mathbf{s}(Cyl(f \times \Delta, g \times \Delta) \sqcup Cyl(f' \times \Delta, g' \times \Delta)) \rightarrow cyl(f \sqcup f', g \sqcup g') .$$

On the other hand, we have that

$$\sigma_{\mathbf{s}} : cyl(f, g) \sqcup cyl(f', g') \rightarrow \mathbf{s}(Cyl(f \times \Delta, g \times \Delta) \sqcup Cyl(f' \times \Delta, g' \times \Delta))$$

is an equivalence, and we are done since  $\sigma_{cyl} = \mathbf{s}\sigma_{Cyl} \circ \sigma_{\mathbf{s}}$ .

Let us prove b). The morphism  $R1 \rightarrow 1$  is in E because of (SDC 5), hence  $1 \sqcup 1 \rightarrow 1$  is an equivalence if and only if  $Id \sqcup Id : R1 \sqcup R1 \rightarrow R1$  is so. First, if in b) we set  $f = f' = 0 : 0 \rightarrow 0$  then  $R1 \sqcup R1 \rightarrow R1$  is just  $\sigma_c : c(0) \sqcup c(0) \rightarrow c(0 \sqcup 0) = c(0)$ .

To see the remaining implication, assume that  $1 \sqcup 1 \rightarrow 1$  is an equivalence. If  $D$  is an object in  $\mathcal{D}$ , denote by  $\rho_D$  the trivial morphism  $D \rightarrow 1$ . Given morphisms  $f : A \rightarrow B$  and  $f' : A' \rightarrow B'$  in  $\mathcal{D}$ , it follows from part a) that

$$\sigma_{cyl} : c(f) \sqcup c(f') = cyl(f, \rho_A) \sqcup cyl(f', \rho_B) \rightarrow cyl(f \sqcup f', \rho_A \sqcup \rho_{A'}) \quad (2.4)$$

is an equivalence. Also, by proposition 2.2.6 we have that the commutative diagram

$$\begin{array}{ccccc} B \sqcup B' & \xleftarrow{f \sqcup f'} & A \sqcup A' & \xrightarrow{\rho_A \sqcup \rho_{A'}} & 1 \sqcup 1 \\ Id \downarrow & & Id \downarrow & & \downarrow \\ B \sqcup B' & \xleftarrow{f \sqcup f'} & A \sqcup A' & \xrightarrow{\rho_A \sqcup \rho_{A'}} & 1, \end{array}$$

gives rise to an equivalence  $cyl(f \sqcup f', \rho_A \sqcup \rho_{A'}) \rightarrow c(f \sqcup f')$  such that composed with (2.4) is just  $\sigma_c$ .  $\square$

**PROPOSITION 2.2.9.** *Consider the following commutative diagram in  $\mathcal{D}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \beta \downarrow \\ Z & \xrightarrow{\alpha} & T. \end{array} \quad (2.5)$$

There exists  $\rho : cyl(f, g) \rightarrow RT$ , natural in (2.5), such that  $\rho I_X = R\alpha$  and  $\rho I_Z = R\beta$ . Visually

$$\begin{array}{ccc}
 RY & \xrightarrow{Rf} & RZ \\
 Rg \downarrow & & \downarrow I_Z \\
 RX & \xrightarrow{I_X} & cyl(f, g) \quad \# \\
 & \searrow \# & \nearrow \rho \\
 & & RT
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \nearrow R\alpha \\
 \searrow R\beta
 \end{array}$$

*Proof.* Just consider the diagram (2.5) in  $\Delta^\circ \mathcal{D}$  through  $- \times \Delta$ , and apply  $\mathbf{s}$  to the morphism  $H : Cyl(f \times \Delta, g \times \Delta) \rightarrow T \times \Delta$  given in proposition 1.5.10.  $\square$

**PROPOSITION 2.2.10.** *Given  $f : X \rightarrow Y$  in  $\mathcal{D}$  then  $f \in E$  if and only if its cone is acyclic, that is, if and only if  $c(f) \rightarrow 1$  is in  $E$ .*

*Proof.* It is enough to apply (SDC 7) to  $f \times \Delta$ , since  $f \in E$  if and only if  $Rf \in E$ .  $\square$

**COROLLARY 2.2.11.** *The class  $E$  is closed under retracts. In other words, if  $g : X' \rightarrow Y'$  is a morphism in  $E$  and there exists a commutative diagram in  $\mathcal{D}$*

$$\begin{array}{ccccc}
 X & \xrightarrow{r} & X' & \xrightarrow{p} & X \\
 f \downarrow & & g \downarrow & & f \downarrow \\
 Y & \xrightarrow{r'} & Y' & \xrightarrow{p'} & Y
 \end{array} \tag{2.6}$$

with  $pr = Id_X$ ,  $p'r' = Id_{Y'}$  (that is,  $f$  is a retract of  $g$ ), then  $f \in E$ .

*Proof.* The image of (2.6) under the cone functor is

$$c(f) \xrightarrow{R} c(g) \xrightarrow{P} c(f)$$

with  $PR = Id_{c(f)}$  in  $\mathcal{D}$ . Since  $g \in E$ , then  $\xi : c(g) \rightarrow 1$  is an equivalence. Hence  $c(f)$  is a retract of an acyclic object, and from lemma 2.1.3 we deduce that  $c(f) \rightarrow 1$  is in  $E$ , so  $f \in E$ .  $\square$

## 2.3 Factorization property of the cylinder functor

This section is devoted to the study of a relevant property of “factorization” satisfied by the functor  $cyl$  in a simplicial descent category  $\mathcal{D}$ . This property

will be very useful in the following section, in fact it is a key point in the development of the relationship between the notions of simplicial descent category and triangulated category.

(2.3.1) Assume given the following commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccccc}
 Z' & \xleftarrow{g'} & X' & \xrightarrow{f'} & Y' \\
 \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow \\
 Z & \xleftarrow{g} & X & \xrightarrow{f} & Y \\
 \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow \\
 Z'' & \xleftarrow{g''} & X'' & \xrightarrow{f''} & Y''
 \end{array} \tag{2.7}$$

Applying the functor  $cyl$  by rows and columns we obtain

$$\begin{aligned}
 cyl(f', g') &\xleftarrow{\delta} cyl(f, g) \xrightarrow{\delta'} cyl(f'', g'') \\
 cyl(\alpha', \alpha) &\xleftarrow{\hat{g}} cyl(\beta', \beta) \xrightarrow{\hat{f}} cyl(\gamma', \gamma).
 \end{aligned}$$

Denote by  $\psi : cyl(R\gamma', R\gamma) \rightarrow cyl(\delta', \delta)$  and  $\psi' : cyl(Rf'', Rg'') \rightarrow cyl(\hat{f}, \hat{g})$  the respective morphisms obtained by applying  $cyl$  to the morphisms in  $\square_1^\circ(\mathcal{D})$ :

$$\begin{array}{ccc}
 RY' & \xleftarrow{R\gamma} & RY \xrightarrow{R\gamma'} RY'' \\
 I \downarrow & & I \downarrow \quad I \downarrow \\
 cyl(f', g') & \xleftarrow{\delta} & cyl(f, g) \xrightarrow{\delta'} cyl(f'', g'')
 \end{array} ; \quad
 \begin{array}{ccc}
 RZ'' & \xleftarrow{Rg''} & RX'' \xrightarrow{Rf''} RY'' \\
 I \downarrow & & I \downarrow \quad I \downarrow \\
 cyl(\alpha', \alpha) & \xleftarrow{\hat{g}} & cyl(\beta', \beta) \xrightarrow{\hat{f}} cyl(\gamma', \gamma)
 \end{array} ,$$

where  $I$  means the corresponding canonical inclusion.

In the same way, denote by  $\hat{\lambda} : cyl(R\gamma', R\gamma) \rightarrow cyl(\gamma', \gamma)$  and  $\tilde{\lambda} : cyl(Rf'', Rg'') \rightarrow cyl(f'', g'')$  the equivalences obtained from

$$\begin{array}{ccc}
 RY' & \xleftarrow{R\gamma} & RY \xrightarrow{R\gamma'} RY'' \\
 \lambda_{Y'} \downarrow & & \lambda_Y \downarrow \quad \lambda_{Y''} \downarrow \\
 Y' & \xleftarrow{\gamma} & Y \xrightarrow{\gamma'} Y''
 \end{array} ; \quad
 \begin{array}{ccc}
 RZ'' & \xleftarrow{Rg''} & RX'' \xrightarrow{Rf''} RY'' \\
 \lambda_{Z''} \downarrow & & \lambda_{X''} \downarrow \quad \lambda_{Y''} \downarrow \\
 Z'' & \xleftarrow{g''} & X'' \xrightarrow{f''} Y''
 \end{array} .$$

**PROPOSITION 2.3.2.** *Under the above notations, the cylinder objects  $cyl(\delta', \delta)$  and  $cyl(\hat{f}, \hat{g})$  are naturally isomorphic in  $Ho\mathcal{D}$ .*

*More concretely, let  $\tilde{T}$  be the simplicial object in  $\mathcal{D}$  obtained by applying  $\widetilde{Cyl}$  to the diagram  $Cyl(\alpha', \alpha) \leftarrow Cyl(\beta', \beta) \rightarrow Cyl(\gamma', \gamma)$ .*

Then there exists isomorphisms  $\Psi : \mathbf{s}(\tilde{T}) \rightarrow \text{cyl}(\delta', \delta)$ ,  $\Phi : \mathbf{s}(\tilde{T}) \rightarrow \text{cyl}(\hat{f}, \hat{g})$  in  $\text{Ho}\mathcal{D}$ , natural in (2.7), such that the diagram

$$\begin{array}{ccccc}
 & R^2Y'' & \xrightarrow{\lambda_{RY''}} & RY'' & \xleftarrow{\lambda_{RY''}} & R^2Y'' \\
 & \swarrow RI & & \swarrow I & & \swarrow I \\
 R\text{cyl}(f'', g'') & \xrightarrow{\lambda} & \text{cyl}(f'', g'') & \xleftarrow{\tilde{\lambda}} & \text{cyl}(Rf'', Rg'') & \\
 \downarrow I & & \downarrow I & & \downarrow I & \\
 & \text{cyl}(R\gamma', R\gamma) & \xrightarrow{\hat{\lambda}} & \text{cyl}(\gamma', \gamma) & \xleftarrow{\lambda} & R\text{cyl}(\gamma', \gamma) \\
 \downarrow \psi & & \downarrow & & \downarrow \psi' & \\
 \text{cyl}(\delta', \delta) & \xleftarrow{\Psi} & \mathbf{s}\tilde{T} & \xrightarrow{\Phi} & \text{cyl}(\hat{f}, \hat{g}) & \\
 & & \uparrow & & \uparrow I & \\
 & & \text{cyl}(R\gamma', R\gamma) & \xrightarrow{\hat{\lambda}} & \text{cyl}(\gamma', \gamma) & \xleftarrow{\lambda} & R\text{cyl}(\gamma', \gamma)
 \end{array} \quad (2.8)$$

commutes in  $\text{Ho}\mathcal{D}$ , and the same holds for  $Z', Z''$  and  $Y'$ .

*Proof.* First of all, note that it is enough to prove the commutativity in  $\text{Ho}\mathcal{D}$  of

$$\begin{array}{ccccc}
 R\text{cyl}(f'', g'') & \xrightarrow{\lambda} & \text{cyl}(f'', g'') & \xleftarrow{\tilde{\lambda}} & \text{cyl}(Rf'', Rg'') \\
 \downarrow I & & \downarrow & & \downarrow \psi' \\
 \text{cyl}(\delta', \delta) & \xleftarrow{\Psi} & \mathbf{s}\tilde{T} & \xrightarrow{\Phi} & \text{cyl}(\hat{f}, \hat{g}) \\
 \uparrow \psi & & \uparrow & & \uparrow I \\
 \text{cyl}(R\gamma', R\gamma) & \xrightarrow{\hat{\lambda}} & \text{cyl}(\gamma', \gamma) & \xleftarrow{\lambda} & R\text{cyl}(\gamma', \gamma)
 \end{array} \quad (2.9)$$

since the remaining squares in diagram (2.8) commutes because of the definitions of the arrows involved in them, or because of the commutativity of the rest of the diagram, together with the fact that the horizontal arrows are isomorphisms in  $\text{Ho}\mathcal{D}$ .

Set  $\text{Cyl}(h, t) = \text{Cyl}(h \times \Delta, t \times \Delta)$  if  $h, t$  are morphisms in  $\mathcal{D}$ .

Consider diagram 2.7 in  $\Delta^\circ\mathcal{D}$  through the functor  $- \times \Delta$  and denote by

$$\begin{array}{ccccc}
 \text{Cyl}(f', g') & \xleftarrow{\rho} & \text{Cyl}(f, g) & \xrightarrow{\rho'} & \text{Cyl}(f'', g'') \\
 \text{Cyl}(\alpha', \alpha) & \xleftarrow{G} & \text{Cyl}(\beta', \beta) & \xrightarrow{F} & \text{Cyl}(\gamma', \gamma)
 \end{array}$$

the result of applying  $\text{Cyl}$  by rows and columns respectively. We also follow the notations introduced in 1.5.14 for  $\phi$  and  $\varphi$ .

Let  $\Theta : \text{Cyl}_{\Delta^\circ\mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G) \rightarrow \text{Cyl}_{\Delta^\circ\mathcal{D}}^{(2)}(\rho \times \Delta, \rho \times \Delta)$  be the canonical



isomorphism in  $\Delta^\circ \Delta^\circ \mathcal{D}$  given in 1.5.15. It is such that the diagram

$$\begin{array}{ccc}
 & \xrightarrow{\phi} Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G) & \xleftarrow{i} \\
 Cyl(f'', g'') \times \Delta & \searrow \Theta \downarrow & \Delta \times Cyl(\gamma', \gamma) \\
 & \xrightarrow{i} Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta) & \xleftarrow{\varphi}
 \end{array} \quad (2.10)$$

commutes. We will compute the image of the above diagram under the functors  $\mathbf{s} \circ \mathbf{D}$ ,  $\mathbf{s} \circ \Delta^\circ \mathbf{s}$  and  $\mathbf{s} \circ \Delta^\circ \mathbf{s} \Gamma$ .

Set  $T = Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G)$  and  $R = Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta)$ . We deduce from 1.7.13 that  $\mathbf{D}T = \widetilde{Cyl}(F, G) = \widetilde{T}$  and  $\mathbf{D}R = \widetilde{Cyl}(\rho', \rho)$ .

Therefore, applying  $\mathbf{s} \circ \mathbf{D}$  to (2.10) we obtain the following commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccc}
 & \xrightarrow{\mathbf{sD}\phi} \widetilde{\mathbf{sCyl}}(F, G) & \xleftarrow{\mathbf{sD}i} \\
 cyl(f'', g'') & \searrow \mathbf{sD}\Theta \downarrow & cyl(\gamma', \gamma) \\
 & \xrightarrow{\mathbf{sD}i} \widetilde{\mathbf{sCyl}}(\rho', \rho) & \xleftarrow{\mathbf{sD}\varphi}
 \end{array} \quad (2.11)$$

where  $\mathbf{sD}\Theta$  is an isomorphism.

Let us compute the image under  $\Delta^\circ \mathbf{s}$  of

$$Cyl(f'', g'') \times \Delta \xrightarrow{\phi} Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times F, \Delta \times G) \xleftarrow{i} \Delta \times Cyl(\gamma', \gamma).$$

It follows from the definitions that

$$(\Delta^\circ \mathbf{s}T)_n = \mathbf{s}(m \rightarrow (Cyl(\gamma', \gamma))_m \sqcup \coprod_{i=1}^n (Cyl(\beta', \beta))_m \sqcup (Cyl(\alpha', \alpha))_m).$$

In addition

$$\begin{aligned}
 (\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta))_n &= \mathbf{s}(m \rightarrow Y'' \sqcup \coprod_{i=1}^n X'' \sqcup Z'') = \mathbf{R}(Y'' \sqcup \coprod_{i=1}^n X'' \sqcup Z'') \\
 (\Delta^\circ \mathbf{s}(\Delta \times Cyl(\gamma', \gamma)))_n &= \mathbf{s}(m \rightarrow Cyl(\gamma', \gamma)_m) = cyl(\gamma', \gamma).
 \end{aligned}$$

By the universal property of the coproduct we have the morphisms

$$\begin{aligned}
 cyl(\gamma', \gamma) \sqcup \coprod_{i=1}^n cyl(\beta', \beta) \sqcup cyl(\alpha', \alpha) &\xrightarrow{\sigma_n} (\Delta^\circ \mathbf{s}T)_n \\
 \mathbf{R}Y'' \sqcup \coprod_{i=1}^n \mathbf{R}X'' \sqcup \mathbf{R}Z'' &\xrightarrow{\sigma_n} \mathbf{R}(Y'' \sqcup \coprod_{i=1}^n X'' \sqcup Z'')
 \end{aligned}$$

in such a way that the following diagram commutes

$$\begin{array}{ccc}
 Cyl(\mathbf{R}f'', \mathbf{R}g'')_n & \xrightarrow{\tilde{\psi}} & Cyl(\mathbf{s}F, \mathbf{s}G)_n \\
 \sigma_n \downarrow & & \sigma_n \downarrow \\
 (\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta))_n & \xrightarrow{(\Delta^\circ \mathbf{s}\phi)_n} & (\Delta^\circ \mathbf{s}T)_n \xleftarrow{(\Delta^\circ \mathbf{s}i)_n} cyl(\gamma', \gamma),
 \end{array}$$

where both morphisms  $\sigma_n$  are equivalences because of (SDC 3), and where  $\mathbf{s}F = \widehat{f}$ ,  $\mathbf{s}G = \widehat{g}$  and  $\mathbf{s}\widetilde{\psi} = \psi'$ . Then, applying  $\mathbf{s}$ , the following diagram commutes

$$\begin{array}{ccc} cyl(\mathbf{R}f'', \mathbf{R}g'') & \xrightarrow{\psi'} & cyl(\widehat{f}, \widehat{g}) \\ \mathbf{s}(\{\sigma_n\}) \downarrow & & \mathbf{s}(\{\sigma_n\}) \downarrow \\ \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta)) & \xrightarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\phi)} & \mathbf{s}(\Delta^\circ \mathbf{s}T) \xleftarrow{\mathbf{s}\Delta^\circ \mathbf{s}i} R_{cyl}(\gamma', \gamma) \end{array} \quad \swarrow I$$

where  $\mathbf{s}(\{\sigma_n\})$  is an equivalence (by the exactness axiom).

On the other hand, the natural transformation  $\mu : \mathbf{s} \circ D \rightarrow \mathbf{s} \circ \Delta^\circ \mathbf{s}$  gives rise to

$$\begin{array}{ccccc} \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta)) & \xrightarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\phi)} & \mathbf{s}(\Delta^\circ \mathbf{s}T) & \xleftarrow{\mathbf{s}\Delta^\circ \mathbf{s}i} & R_{cyl}(\gamma', \gamma) \\ \uparrow \mu_{Cyl(f'', g'') \times \Delta} & & \uparrow \mu_T & & \uparrow \mu_{\Delta \times Cyl(\gamma', \gamma)} \\ cyl(f'', g'') & \xrightarrow{\mathbf{s}D\phi} & \widetilde{\mathbf{s}Cyl}(F, G) & \xleftarrow{\mathbf{s}Di} & cyl(\gamma', \gamma) \end{array}$$

From the equations (2.1) describing the compatibility between  $\lambda$  and  $\mu$  we deduce that the following diagram commutes in  $Ho\mathcal{D}$

$$\begin{array}{ccccc} \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta)) & \xrightarrow{\mathbf{s}(\Delta^\circ \mathbf{s}i)} & \mathbf{s}(\Delta^\circ \mathbf{s}T) & \xleftarrow{\mathbf{s}\Delta^\circ \mathbf{s}i} & R_{cyl}(\gamma', \gamma) \\ \mathbf{s}(\{\lambda_n\}) \downarrow & & \uparrow \mu_T & & \downarrow \lambda_{cyl(\gamma', \gamma)} \\ cyl(f'', g'') & \xrightarrow{\mathbf{s}D\phi} & \widetilde{\mathbf{s}Cyl}(F, G) & \xleftarrow{\mathbf{s}Di} & cyl(\gamma', \gamma) \end{array}$$

If we join the two resulting diagrams, we obtain

$$\begin{array}{ccccc} cyl(\mathbf{R}f'', \mathbf{R}g'') & \xrightarrow{\psi'} & cyl(\widehat{f}, \widehat{g}) & \xleftarrow{I} & R_{cyl}(\gamma', \gamma) \\ \mathbf{s}(\{\sigma_n\}) \downarrow & & \mathbf{s}(\{\sigma_n\}) \downarrow & & \downarrow \lambda_{cyl(\gamma', \gamma)} \\ \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(f'', g'') \times \Delta)) & & \mathbf{s}(\Delta^\circ \mathbf{s}T) & & \\ \mathbf{s}(\{\lambda_n\}) \downarrow & & \uparrow \mu_T & & \\ cyl(f'', g'') & \xrightarrow{\mathbf{s}D\phi} & \widetilde{\mathbf{s}Cyl}(F, G) & \xleftarrow{\mathbf{s}Di} & cyl(\gamma', \gamma) \end{array}$$

that is just the right side of (2.9) taking  $\Phi = \mathbf{s}(\{\sigma_n\})^{-1} \circ \mu_T$ , and noting that  $\mathbf{s}(\{\lambda_n \circ \sigma_n\}) = \widetilde{\lambda}$ .

Indeed, it follows from the compatibility (2.2) between  $\lambda$  and  $\sigma$  that the composition

$$RY'' \sqcup \coprod_{n=1}^n RX'' \sqcup RZ'' \xrightarrow{\sigma_n} R(Y'' \sqcup \coprod_{n=1}^n X'' \sqcup Z'') \xrightarrow{\lambda_n} Y'' \sqcup \coprod_{n=1}^n X'' \sqcup Z''$$

is equal to  $\lambda_{Y''} \sqcup \coprod^n \lambda_{X''} \sqcup \lambda_{Z''}$ .

It remains to see the existence of the left side of (2.9). In order to do that, we argue in a similar way to compute the image under  $\Delta^\circ \mathbf{s}\Gamma$  of

$$Cyl(f'', g'') \times \Delta \xrightarrow{i} Cyl_{\Delta^\circ \mathcal{D}}^{(2)}(\rho' \times \Delta, \rho \times \Delta) \xleftarrow{\varphi} \Delta \times Cyl(\gamma', \gamma) .$$

By definition

$$\begin{aligned} (\Delta^\circ \mathbf{s}\Gamma R)_n &= \mathbf{s}(m \rightarrow Cyl(f'', g''))_m \sqcup \coprod^n Cyl(f, g)_m \sqcup Cyl(f', g')_m \\ (\Delta^\circ \mathbf{s}\Gamma(Cyl(f'', g'') \times \Delta))_n &= (\Delta^\circ \mathbf{s}(\Delta \times Cyl(f'', g'')))_n = \mathbf{s}(m \rightarrow Cyl(f'', g''))_m \\ (\Delta^\circ \mathbf{s}\Gamma(\Delta \times Cyl(\gamma', \gamma)))_n &= (\Delta^\circ \mathbf{s}Cyl(\gamma', \gamma) \times \Delta)_n = \mathbf{s}(m \rightarrow Y'' \sqcup \coprod^n Y \sqcup Y') . \end{aligned}$$

Again by the universal property of the coproduct we have the following commutative diagram

$$\begin{array}{ccccc} cyl(f'', g'') & \xrightarrow{(\Delta^\circ \mathbf{s}\Gamma i)_n} & (\Delta^\circ \mathbf{s}\Gamma R)_n & \xleftarrow{(\Delta^\circ \mathbf{s}\Gamma \varphi)_n} & (\Delta^\circ \mathbf{s}(Cyl(\gamma', \gamma) \times \Delta))_n \\ & \searrow i & \uparrow \sigma_n & & \uparrow \sigma_n \\ & & Cyl(\mathbf{s}\rho', \mathbf{s}\rho)_n & \xleftarrow{\bar{\psi}_n} & cyl(\gamma', \gamma) , \end{array}$$

where  $\mathbf{s}\rho' = \delta'$ ,  $\mathbf{s}\rho = \delta$  and  $\mathbf{s}\bar{\psi} = \psi$ . Hence applying  $\mathbf{s}$  we obtain

$$\begin{array}{ccccc} Rcyl(f'', g'') & \xrightarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\Gamma i)} & \mathbf{s}(\Delta^\circ \mathbf{s}\Gamma R) & \xleftarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\Gamma \varphi)} & \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(\gamma', \gamma) \times \Delta)) \\ & \searrow I & \uparrow \mathbf{s}(\{\sigma_n\}) & & \uparrow \mathbf{s}(\{\sigma_n\}) \\ & & cyl(\delta', \delta) & \xleftarrow{\psi} & cyl(\gamma', \gamma) . \end{array}$$

The natural transformation  $\mu\Gamma : \mathbf{s}\mathcal{D} \rightarrow \mathbf{s}_\circ \Delta^\circ \mathbf{s}_\circ \Gamma$  gives rise to

$$\begin{array}{ccccc} cyl(f'', g'') & \xrightarrow{I} & \widetilde{Cyl}(\rho', \rho) & \xleftarrow{\mathbf{s}\mathcal{D}\varphi} & cyl(\gamma', \gamma) \\ \downarrow \mu_{\Delta \times Cyl(f'', g'')} & & \downarrow \mu_{\Gamma R} & & \downarrow \mu_{Cyl(\gamma', \gamma) \times \Delta} \\ Rcyl(f'', g'') & \xrightarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\Gamma i)} & \mathbf{s}(\Delta^\circ \mathbf{s}\Gamma R) & \xleftarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\Gamma \varphi)} & \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(\gamma', \gamma) \times \Delta)) , \end{array}$$

where we can replace  $\mu_{\Delta \times Cyl(f'', g'')}$  by  $\lambda_{cyl(f'', g'')}$  and  $\mu_{Cyl(\gamma', \gamma) \times \Delta}$  by  $\mathbf{s}(\{\lambda_n\})$ .

Putting all together we get

$$\begin{array}{ccccc} cyl(f'', g'') & \xrightarrow{I} & \widetilde{Cyl}(\rho', \rho) & \xleftarrow{\mathbf{s}\mathcal{D}\varphi} & cyl(\gamma', \gamma) \\ \uparrow \lambda_{cyl(f'', g'')} & & \downarrow \mu_{\Gamma R} & & \uparrow \mathbf{s}(\{\lambda_n\}) \\ & & \mathbf{s}(\Delta^\circ \mathbf{s}\Gamma R) & \xleftarrow{\mathbf{s}(\Delta^\circ \mathbf{s}\Gamma \varphi)} & \mathbf{s}(\Delta^\circ \mathbf{s}(Cyl(\gamma', \gamma) \times \Delta)) \\ & & \uparrow \mathbf{s}(\{\sigma_n\}) & & \uparrow \mathbf{s}(\{\sigma_n\}) \\ Rcyl(f'', g'') & \xrightarrow{I} & cyl(\delta', \delta) & \xleftarrow{\psi} & cyl(\gamma', \gamma) . \end{array}$$

Again,  $\mathbf{s}(\{\lambda_n\} \circ \mathbf{s}(\{\sigma_n\})) = \widehat{\lambda}$ . Then, adjoining (2.11), the result is

$$\begin{array}{ccccc}
cyl(f'', g'') & \xrightarrow{\mathbf{sD}\phi} & \widetilde{Cyl}(F, G) & \xleftarrow{\mathbf{sDi}} & cyl(\gamma', \gamma) \\
\uparrow Id & & \downarrow \mathbf{sD}\Theta & & \uparrow Id \\
cyl(f'', g'') & \xrightarrow{I} & \widetilde{Cyl}(\rho', \rho) & \xleftarrow{\mathbf{sD}\varphi} & cyl(\gamma', \gamma) \\
\uparrow \lambda_{cyl(f'', g'')} & & \downarrow \mu_{\Gamma R} & & \uparrow \widehat{\lambda} \\
Rcyl(f'', g'') & \xrightarrow{I} & \mathbf{s}(\Delta^\circ \mathbf{s}\Gamma R) & \xleftarrow{\psi} & cyl(\gamma', \gamma) \\
& & \uparrow \mathbf{s}(\{\sigma_n\}) & & \\
& & & & 
\end{array}$$

To finish just take  $\Psi = \mathbf{s}(\{\sigma_n\})^{-1} \circ \mu_{\Gamma R} \circ \mathbf{sD}\Theta$ .  $\square$

In order to deduce from this result an analogous property for the cone functor, we need the following lemma.

**LEMMA 2.3.3.** *Let  $Z \xleftarrow{g} X \xrightarrow{f} Y$  be a diagram of simplicial objects in  $\mathcal{D}$ . There exists an isomorphism  $\Phi : \mathbf{s}(\widetilde{Cyl}(f, g)) \rightarrow cyl(\mathbf{s}f, \mathbf{s}g)$  in  $Ho\mathcal{D}$ , natural in  $f$  and  $g$ , that fits into the following commutative diagram of  $Ho\mathcal{D}$*

$$\begin{array}{ccc}
\mathbf{s}Y & \xrightarrow{\mu_{\Delta \times Y}} & R(\mathbf{s}Y) \\
& \searrow sj_Y & \searrow I_{\mathbf{s}Y} \\
& & \mathbf{s}(\widetilde{Cyl}(f, g)) \xrightarrow{\Phi} cyl(\mathbf{s}f, \mathbf{s}g) \\
& \nearrow sj_Z & \nearrow I_{\mathbf{s}Z} \\
\mathbf{s}Z & \xrightarrow{\mu_{\Delta \times Z}} & R(\mathbf{s}Z)
\end{array} \tag{2.12}$$

where  $j_Y, j_Z$  are the canonical inclusions given in 1.7.5.

*Proof.* This result is a consequence of the factorization, additivity and exactness axioms, together with 1.7.13.

Indeed, consider  $(\Delta \times f, \Delta \times g) \in \Omega^{(1)}(\Delta^\circ \mathcal{D})$ .

Then  $T = Cyl_{\Delta^\circ \mathcal{D}}^{(1)}(\Delta \times f, \Delta \times g)$  is a bisimplicial object in  $\mathcal{D}$  whose diagonal is just  $\widetilde{Cyl}(f, g)$ .

By the factorization axiom,  $\mu_T : \mathbf{s}(DT) \rightarrow \mathbf{s} \circ \Delta^\circ \mathbf{s}(T) = \mathbf{s}(n \rightarrow \mathbf{s}(m \rightarrow T_{n,m}))$  is an equivalence. The simplicial object  $\Delta^\circ \mathbf{s}(T)$  is given in degree  $n$  by

$$(\Delta^\circ \mathbf{s}(T))_n = \mathbf{s}(m \rightarrow Y_m \sqcup X_m^{(n-1)} \sqcup \dots \sqcup X_m^{(0)} \sqcup Z_m) .$$

Hence by (SDC 3) we have an equivalence

$$\sigma_n : \mathbf{s}Y \sqcup (\mathbf{s}X)^{(n-1)} \sqcup \dots \sqcup (\mathbf{s}X)^{(0)} \sqcup \mathbf{s}Z \longrightarrow (\Delta^\circ \mathbf{s}(T))_n .$$

Moreover,  $\mathbf{s}Y \sqcup (\mathbf{s}X)^{(n-1)} \sqcup \dots \sqcup (\mathbf{s}X)^{(0)} \sqcup \mathbf{s}Z = \text{Cyl}((\mathbf{s}f) \times \Delta, (\mathbf{s}g) \times \Delta)_n$  and since  $\sigma_n$  is obtained by the universal property of the coproduct, we have that the following diagram commutes

$$\begin{array}{ccc}
 \mathbf{s}Y & \xrightarrow{\quad \mathbf{s}(i_Y) \quad} & (\Delta^\circ \mathbf{s}(T))_n \\
 \searrow i_{\mathbf{s}Y} & & \uparrow i_{\mathbf{s}Z} \\
 & \text{Cyl}((\mathbf{s}f) \times \Delta, (\mathbf{s}g) \times \Delta)_n \xrightarrow{\sigma_n} & \\
 \nearrow i_{\mathbf{s}Z} & & \\
 \mathbf{s}Z & \xrightarrow{\quad \mathbf{s}(i_Z) \quad} & 
 \end{array}$$

By the naturality of  $\sigma$ , we get the morphism between simplicial objects  $\varrho = \{\sigma_n\}_n : \text{Cyl}((\mathbf{s}f) \times \Delta, (\mathbf{s}g) \times \Delta) \rightarrow \Delta^\circ \mathbf{s}(T)$ .

Applying  $\mathbf{s}$  we obtain the commutative diagram

$$\begin{array}{ccc}
 \mathbf{R}(\mathbf{s}Y) & \xrightarrow{\quad \mathbf{s}(\mathbf{s}(i_Y)) \quad} & \mathbf{s} \circ \Delta^\circ \mathbf{s}(T) \\
 \searrow I_{\mathbf{s}Y} & & \uparrow I_{\mathbf{s}Z} \\
 & \text{cyl}(\mathbf{s}f, \mathbf{s}g) \xrightarrow{\mathbf{s}\varrho} & \\
 \nearrow I_{\mathbf{s}Z} & & \\
 \mathbf{R}(\mathbf{s}Z) & \xrightarrow{\quad \mathbf{s}(\mathbf{s}(i_Z)) \quad} & 
 \end{array}$$

where  $\mathbf{s}\varrho$  is an equivalence by the exactness axiom.

Finally,  $\mathbf{R}(\mathbf{s}Y) = \mathbf{s} \circ \Delta^\circ \mathbf{s}(\Delta \times Y)$ , and from the naturality of  $\mu$  follows that the following diagram commutes

$$\begin{array}{ccccc}
 \mathbf{s}Y & \xrightarrow{\mu_{\Delta \times Y}} & \mathbf{R}(\mathbf{s}Y) & \xrightarrow{I_{\mathbf{s}Y}} & \mathbf{s} \circ \Delta^\circ \mathbf{s}(T) \\
 \searrow \mathbf{s}(j_Y) & & \searrow & & \uparrow \mathbf{s}(j_Z) \\
 & \widetilde{\mathbf{s}(\text{Cyl}(f, g))} & \xrightarrow{\mu_T} & & \\
 \nearrow \mathbf{s}(j_Z) & & \nearrow & & \\
 \mathbf{s}Z & \xrightarrow{\mu_{\Delta \times Z}} & \mathbf{R}(\mathbf{s}Z) & \xrightarrow{I_{\mathbf{s}Z}} & 
 \end{array}$$

Therefore, it suffices to take  $\Phi = (\mathbf{s}\varrho)^{-1} \circ \mu_T : \widetilde{\mathbf{s}(\text{Cyl}(f, g))} \rightarrow \text{cyl}(\mathbf{s}f, \mathbf{s}g)$ .  $\square$

(2.3.4) Consider the following commutative square of  $\mathcal{D}$

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 g \downarrow & & \downarrow g' \\
 X' & \xrightarrow{f'} & Y'
 \end{array} \tag{2.13}$$

Let  $\widehat{g} : c(f) \rightarrow c(f')$  and  $\widehat{f} : c(g) \rightarrow c(g')$  be the morphisms deduced from the functoriality of the cone, as well as  $\psi : c(\mathbf{R}f') \rightarrow c(\widehat{f})$ ,  $\psi' : c(\mathbf{R}g') \rightarrow c(\widehat{g})$ ,

$\tilde{\lambda} : c(Rf') \rightarrow c(f)$  and  $\hat{\lambda} : c(Rg') \rightarrow c(g)$  those obtained from the following commutative diagrams

$$\begin{array}{ccc} RX' \xrightarrow{Rf'} RY' & & RY \xrightarrow{Rg'} RY' \\ I \downarrow & I \downarrow & \\ c(g) \xrightarrow{\hat{f}} c(g') & & c(f) \xrightarrow{\hat{g}} c(f') \end{array} ; \quad \begin{array}{ccc} RX' \xrightarrow{Rf'} RY' & & RY \xrightarrow{Rg'} RY' \\ \lambda_{X'} \downarrow & \lambda_{Y'} \downarrow & \\ X' \xrightarrow{f'} Y' & & Y \xrightarrow{g'} Y' \end{array} ,$$

where each  $I$  denotes the corresponding canonical inclusion.

**COROLLARY 2.3.5.** *Under the above notations, the cone objects  $c(\hat{f})$  and  $c(\hat{g})$  are naturally isomorphic in  $Ho\mathcal{D}$ .*

*If  $\tilde{T} \in \Delta^\circ \mathcal{D}$  is  $\widetilde{Cyl}$  of the diagram  $1 \times \Delta \leftarrow C(g) \rightarrow C(g')$ , then there exists isomorphisms  $\Psi : s(\tilde{T}) \rightarrow c(\hat{g})$  and  $\Phi : s(\tilde{T}) \rightarrow c(\hat{f})$  in  $Ho\mathcal{D}$ , natural in (2.13), such that the diagram*

$$\begin{array}{ccccc} & R^2Y' & \xrightarrow{\lambda_{RY'}} & RY' & \xleftarrow{\lambda_{RY'}} R^2Y' \\ & \swarrow RI & & \swarrow I & \swarrow I \\ Rc(f') & \xrightarrow{\lambda} & c(f') & \xleftarrow{\tilde{\lambda}} & c(Rf') \\ \downarrow I & & \downarrow I & & \downarrow I \\ c(Rg') & \xrightarrow{\hat{\lambda}} & c(g') & \xleftarrow{\lambda} & Rc(g') \\ \swarrow \psi' & & \downarrow \eta & & \downarrow \psi \\ c(\hat{g}) & \xleftarrow{\Psi} & s\tilde{T} & \xrightarrow{\Phi} & c(\hat{f}) \end{array}$$

*commutes in  $Ho\mathcal{D}$ . The map  $\eta : c(f') \rightarrow s\tilde{T}$  is the simple of the morphism induced by the inclusions of  $Y'$  and  $X'$  into  $C(g)$  and  $C(g')$  respectively, whereas  $\eta' : c(g') \rightarrow s\tilde{T}$  is just the simple of the inclusion of  $C(g')$  into  $\tilde{T}$ .*

*Proof.* Again, it suffices to prove the commutativity of

$$\begin{array}{ccccc} Rc(f') & \xrightarrow{\lambda} & c(f') & \xleftarrow{\tilde{\lambda}} & c(Rf') \\ \downarrow I & & \downarrow \eta & & \downarrow \psi \\ c(\hat{g}) & \xleftarrow{\Psi} & s\tilde{T} & \xrightarrow{\Phi} & c(\hat{f}) \\ \uparrow \psi' & & \uparrow \eta' & & \uparrow I \\ c(Rg') & \xrightarrow{\hat{\lambda}} & c(g') & \xleftarrow{\lambda} & Rc(g') \end{array}$$

Let us see that this is true for the left side of the diagram, since the commutativity of the right side can be checked similarly.

We complete diagram (2.13) to

$$\begin{array}{ccccc}
 1 & \longleftarrow & 1 & \longrightarrow & 1 \\
 \uparrow & & \uparrow & & \uparrow v \\
 1 & \longleftarrow & X & \xrightarrow{f} & Y \\
 \downarrow & & g \downarrow & & \downarrow g' \\
 1 & \longleftarrow & X' & \xrightarrow{f'} & Y'.
 \end{array}$$

The image under  $cyl$  of the rows of this diagram is

$$cyl(1) \xleftarrow{\varrho} c(f) \xrightarrow{\widehat{g}} c(f').$$

Denote by  $\widetilde{\psi} : cyl(Rg', Rv) \rightarrow cyl(\widehat{g}, \varrho)$  and  $\overline{\lambda} : cyl(Rg', Rv) \rightarrow c(g')$  the result of applying  $cyl$  to

$$\begin{array}{ccccc}
 R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\
 I \downarrow & & I \downarrow & & I \downarrow \\
 cyl(1) & \xleftarrow{\varrho} & c(f) & \xrightarrow{\widehat{g}} & c(f')
 \end{array}
 \quad ; \quad
 \begin{array}{ccccc}
 R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\
 \lambda_1 \downarrow & & \lambda_Y \downarrow & & \lambda_{Y'} \downarrow \\
 1 & \xleftarrow{v} & Y & \xrightarrow{g'} & Y'.
 \end{array}$$

By the last proposition we have that if  $\widehat{T}$  is  $\widetilde{Cyl}$  of the diagram  $Cyl(1 \times \Delta) \leftarrow C(g) \rightarrow C(g')$  then we have an isomorphism  $\Psi'$  in  $Ho\mathcal{D}$  such that the following diagram commutes

$$\begin{array}{ccccc}
 Rc(f') & \xrightarrow{I} & cyl(\widehat{g}, \varrho) & \xleftarrow{\widetilde{\psi}} & cyl(Rg', Rv) \\
 \lambda \downarrow & & \Psi' \uparrow & & \overline{\lambda} \downarrow \\
 c(f') & \xrightarrow{\widehat{\eta}} & s\widehat{T} & \xleftarrow{\widehat{\eta}'} & c(g'),
 \end{array}$$

where  $\widehat{\eta}'$  is the simple of the canonical inclusion of  $C(g')$  into  $\widehat{T}$ , whereas  $\eta$  is the simple of the morphism induced by the inclusions of  $Y'$ ,  $X'$  and  $1 \times \Delta$  into  $C(g)$ ,  $C(g')$  and  $Cyl(1 \times \Delta)$  respectively.

If  $R1 \xrightarrow{J_1} cyl(1) \xleftarrow{I_1} R1$  are the canonical inclusions obtained by applying  $cyl$  to  $1 \leftarrow 1 \rightarrow 1$ , by 2.2.9, we deduce the existence of a morphism  $\rho : cyl(1) \rightarrow R1$  with  $\rho I_1 = \rho J_1 = Id_{R1}$ .

Since  $Id : 1 \rightarrow 1$  is an equivalence, it follows from the acyclicity axiom that  $J_1$  is so, and because of the 2 out of 3 property we get that  $\rho$  is in E.

Therefore  $\rho' = \lambda_1 \rho : cyl(1) \rightarrow 1$  is also an equivalence, and the morphism of cubical diagrams

$$\begin{array}{ccccc} cyl(1) & \xleftarrow{\varrho} & c(f) & \xrightarrow{\widehat{g}} & c(f') \\ \rho' \downarrow & & Id \downarrow & & Id \downarrow \\ 1 & \xleftarrow{\quad} & c(f) & \xrightarrow{\widehat{g}} & c(f') \end{array}$$

gives rise to the equivalence  $\tau : cyl(\widehat{g}, \varrho) \rightarrow c(\widehat{g})$  such that the following diagram commutes

$$\begin{array}{ccc} cyl(\widehat{g}, \varrho) & \xleftarrow{I} & Rc(f') \\ \tau \downarrow & & Id \downarrow \\ c(\widehat{g}) & \xleftarrow{I} & Rc(f') . \end{array}$$

On the other hand, the trivial morphism  $\lambda_1 : R1 \rightarrow 1$  induces

$$\begin{array}{ccccc} R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\ \lambda_1 \downarrow & & Id \downarrow & & Id \downarrow \\ 1 & \xleftarrow{\quad} & RY & \xrightarrow{Rg'} & RY' \end{array}$$

and applying  $cyl$ , we get an equivalence  $\tau' : cyl(Rg', Rv) \rightarrow c(Rg')$ .

Hence, the following diagram is also commutative

$$\begin{array}{ccccc} Rc(f') & \xrightarrow{I} & c(\widehat{g}) & \xleftarrow{\psi'} & c(Rg') \\ Id \uparrow & & \tau \uparrow & & \tau' \uparrow \\ Rc(f') & \xrightarrow{I} & cyl(\widehat{g}, \varrho) & \xleftarrow{\widetilde{\psi}} & cyl(Rg', Rv) \\ \lambda \downarrow & & \Psi' \uparrow & & \overline{\lambda} \downarrow \\ c(f') & \xrightarrow{\widehat{\eta}} & s\widehat{T} & \xleftarrow{\widehat{\eta}'} & c(g') . \end{array}$$

Indeed,  $\tau \circ \widetilde{\psi} = \psi' \circ \tau'$  since both morphisms are respectively the image under  $cyl$  of the morphism in  $\square_1 \mathcal{D}$  given by these two compositions

$$\begin{array}{ccccc} R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\ I_1 \downarrow & & I \downarrow & & I \downarrow \\ cyl(1) & \xleftarrow{\varrho} & c(f) & \xrightarrow{\widehat{g}} & c(f') \\ \downarrow & & Id \downarrow & & Id \downarrow \\ 1 & \xleftarrow{\quad} & c(f) & \xrightarrow{\widehat{g}} & c(f') \end{array} \quad ; \quad \begin{array}{ccccc} R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\ \downarrow & & Id \downarrow & & Id \downarrow \\ 1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\ \downarrow & & I \downarrow & & I \downarrow \\ 1 & \xleftarrow{\quad} & c(f) & \xrightarrow{\widehat{g}} & c(f') . \end{array}$$



Moreover, it holds in  $Ho\mathcal{D}$  that  $\bar{\lambda}_\circ(\tau')^{-1} = \hat{\lambda}$  because in  $\mathcal{D}$ ,  $\hat{\lambda}_\circ\tau'$  is *cyl* of the composition

$$\begin{array}{ccccc} R1 & \xleftarrow{Rv} & RY & \xrightarrow{Rg'} & RY' \\ \lambda_1 \downarrow & & Id \downarrow & & Id \downarrow \\ 1 & \xleftarrow{\quad} & RY & \xrightarrow{Rg'} & RY' \\ Id \downarrow & & \lambda_Y \downarrow & & \lambda_{Y'} \downarrow \\ 1 & \xleftarrow{\quad} & Y & \xrightarrow{g'} & Y' . \end{array}$$

that agrees with  $\bar{\lambda}$  by definition.

Hence, setting  $\Psi'' = \tau_\circ\Psi'$  we have the following commutative diagram

$$\begin{array}{ccccc} Rc(f') & \xrightarrow{I} & c(\hat{g}) & \xleftarrow{\psi'} & c(Rg') \\ \lambda \downarrow & & \Psi'' \uparrow & & \hat{\lambda} \downarrow \\ c(f') & \xrightarrow{\hat{\eta}} & \mathbf{s}\hat{T} & \xleftarrow{\hat{\eta}'} & c(g') . \end{array} \quad (2.14)$$

and it remains to show that

$$\hat{T} = \widetilde{Cyl}(Cyl(1 \times \Delta) \leftarrow C(g) \rightarrow C(g')) \text{ and } \tilde{T} = \widetilde{Cyl}(1 \times \Delta \leftarrow C(g) \rightarrow C(g'))$$

are such that  $\mathbf{s}(\hat{T})$  and  $\mathbf{s}(\tilde{T})$  are naturally equivalent.

Indeed, if  $\nu : Cyl(1 \times \Delta) \rightarrow 1 \times \Delta$  is the morphisms deduced from 1.5.10, then the diagram

$$\begin{array}{ccccc} Cyl(1 \times \Delta) & \longleftarrow & C(g) & \longrightarrow & C(g') \\ \nu \downarrow & & Id \downarrow & & Id \downarrow \\ 1 \times \Delta & \longleftarrow & C(g) & \longrightarrow & C(g') \end{array}$$

gives rise (by applying  $\widetilde{Cyl}$ ) to the morphism  $\vartheta : \hat{T} \rightarrow \tilde{T}$  between simplicial objects. The image under  $\mathbf{s}$  of the above diagram produces the morphism in  $\square_1^c\mathcal{D}$  consisting of

$$\begin{array}{ccccc} cyl(1) & \xleftarrow{\varrho'} & c(g) & \xrightarrow{\hat{f}} & c(g') \\ s\nu \downarrow & & Id \downarrow & & Id \downarrow \\ R1 & \xleftarrow{\nu'} & c(g) & \xrightarrow{\hat{f}} & c(g') \end{array}$$

and such that  $s\nu \in E$ , since  $cyl(1) = c(Id_1)$  and  $R1$  are equivalent to 1. Then, by 2.2.6 we have that the induced morphism  $\vartheta' : cyl(\hat{f}, s\varrho') \rightarrow cyl(\hat{f}, \nu')$  is an equivalence.

Finally, from lemma 2.3.3 we deduce the commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccc} \widehat{\mathbf{s}T} & \xrightarrow{\widehat{\Phi}} & \text{cyl}(\widehat{f}, \varrho') \\ \text{s}\vartheta \downarrow & & \downarrow \vartheta' \\ \widetilde{\mathbf{s}T} & \xrightarrow{\widetilde{\Phi}} & \text{cyl}(\widehat{f}, \nu') . \end{array}$$

It follows that  $\text{s}\vartheta : \widehat{\mathbf{s}T} \rightarrow \widetilde{\mathbf{s}T}$  is an equivalence. Consequently, by the definition of  $\vartheta$  it is clear that, after adjoining this morphism to (2.14), we get the desired commutative diagram

$$\begin{array}{ccccc} \text{Rc}(f') & \xrightarrow{I} & c(\widehat{g}) & \xleftarrow{\psi'} & c(\text{R}g') \\ \lambda \downarrow & & \uparrow \Psi'' & & \downarrow \widehat{\lambda} \\ c(f') & \xrightarrow{\widehat{\eta}} & \widehat{\mathbf{s}T} & \xleftarrow{\widehat{\eta}'} & c(g') \\ & \searrow \eta & \downarrow \text{s}\vartheta & \swarrow \eta' & \\ & & \widetilde{\mathbf{s}T} & & \end{array}$$

□

## 2.4 Acyclicity criterion for the cylinder functor

In this section we develop a generalization of the acyclicity axiom of the notion of simplicial descent category.

More concretely, the question is if (SDC 7) remains true when we consider any augmentation  $X \rightarrow X_{-1} \times \Delta$  instead of  $X \rightarrow 1 \times \Delta$ .

The answer is affirmative for the “only if” part of (SDC 7), whereas the other part will be true under certain extra hypothesis.

The acyclicity criterion is necessary in the next chapter in order to establish the “transfer lemma”, and it will play a crucial role in the study of  $Ho\mathcal{D}$ .

**PROPOSITION 2.4.1.** *Consider morphisms  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\mathcal{D}$ . The functor  $\text{cyl}$  gives rise to the diagram*

$$\begin{array}{ccc} \text{R}X & \xrightarrow{\text{R}f} & \text{R}Y \\ \text{R}g \downarrow & & \downarrow I_Y \\ \text{R}Z & \xrightarrow{I_Z} & \text{cyl}(f, g) \end{array}$$

that satisfies the following properties

**a)**  $f \in \text{E}$  if and only if  $I_Z$  is in  $\text{E}$ .

**b)**  $g \in E$  if and only if  $I_Y$  is in  $E$ .

In addition, the functor  $cyl'$  verifies the analogous properties.

*Proof.* Consider the commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccccc}
 1 & \longleftarrow & 0 & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \uparrow \\
 Z & \longleftarrow & 0 & \longrightarrow & 0 \\
 \downarrow Id & & \downarrow & & \downarrow \\
 Z & \xleftarrow{g} & X & \xrightarrow{f} & Y .
 \end{array}$$

If we apply  $cyl$  by rows and columns we get

$$\begin{array}{ccccc}
 R1 & \xleftarrow{\varrho} & RZ & \xrightarrow{I_Z} & cyl(f, g) \\
 c(Id_Z) & \xleftarrow{\varrho'} & RX & \xrightarrow{Rf} & RY .
 \end{array}$$

By the factorization property of the cylinder functor, 2.3.2, we deduce that  $cyl(I_Z, \varrho)$  and  $cyl(Rf, \varrho')$  are isomorphic in  $Ho\mathcal{D}$ .

Since  $Id_Z \in E$ , then  $c(Id_Z) \rightarrow 1$  is an equivalence, and therefore the morphism  $cyl(Rf, \varrho') \rightarrow c(Rf)$  obtained by applying  $cyl$  to

$$\begin{array}{ccccc}
 c(Id_Z) & \xleftarrow{\varrho'} & RX & \xrightarrow{Rf} & RY \\
 \downarrow & & \downarrow Id & & \downarrow Id \\
 1 & \xleftarrow{\quad} & RX & \xrightarrow{Rf} & RY
 \end{array}$$

is an equivalence.

On the other hand, it follows from the normalization axiom that  $R1 \rightarrow 1$  is in  $E$ , and arguing as before we obtain an equivalence  $cyl(I_Z, \varrho) \rightarrow c(I_Z)$ .

Therefore  $c(Rf)$  is isomorphic to  $c(I_Z)$  in  $Ho\mathcal{D}$ , so

$$c(Rf) \rightarrow 1 \text{ is an equivalence if and only if } c(I_Z) \rightarrow 1 \text{ is so.}$$

Then, by 2.2.10,  $I_Z$  is an equivalence if and only if  $Rf$  is so, and by (SDC 4) this happens if and only if  $f \in E$ .

The similar result for  $cyl'$  follows from the symmetry axiom. Indeed,

$$cyl'(f, g) = sCyl'(f \times \Delta, g \times \Delta) = s\Upsilon Cyl \Upsilon(f \times \Delta, g \times \Delta) = s\Upsilon Cyl(f \times \Delta, g \times \Delta),$$

since  $\Upsilon(h \times \Delta) = h$  for every morphism  $h$  in  $\mathcal{D}$ .

Hence  $I_Z = s(i_Z) \in E$  if and only if  $s(\Upsilon i_Z) : RZ \rightarrow cyl'(f, g)$  is in  $E$ , but this

morphism is just the canonical inclusion of  $RZ$  into  $cyl'(f, g)$ , so we get a).

To see b), from the statement a) for  $cyl'$  we get that  $g \in E$  if and only if the inclusion  $RY \rightarrow cyl'(g, f)$  is so. From lemma 2.3.3 we deduce the existence of a morphism  $\tau : cyl'(g, f) \rightarrow cyl(f, g)$  such that the following diagram commutes

$$\begin{array}{ccccc} RY & & & & \\ & \searrow & & \nearrow & \\ & cyl'(g, f) & \xrightarrow{\tau} & cyl(f, g) & \\ & \nearrow & & \searrow & \\ RZ & & & & \end{array}$$

Then  $RY \rightarrow cyl'(g, f)$  is an equivalence if and only if  $RY \rightarrow cyl(f, g)$  is so, that finish the proof of b).

The statement b) for  $cyl'$  can be proved analogously using the symmetry axiom.  $\square$

**LEMMA 2.4.2.** *Consider the morphisms  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\Delta^\circ \mathcal{D}$ . The functors  $\widetilde{Cyl}$  and  $s$  give rise to a diagram in  $\mathcal{D}$*

$$\begin{array}{ccc} sX & \xrightarrow{sf} & sY \\ sg \downarrow & & \downarrow sj_Y \\ sZ & \xrightarrow{sj_Z} & s\widetilde{Cyl}(f, g) \end{array} \quad (2.15)$$

such that

**a)**  $sf \in E$  if and only if  $sj_Z$  is in  $E$ .

**b)**  $sg \in E$  if and only if  $sj_Y$  is in  $E$ .

*Proof.* First, assume that  $X, Y$  and  $Z$  are objects in  $\mathcal{D}$ . In this case, by 2.2.5 a),  $s\widetilde{Cyl}(f \times \Delta, g \times \Delta) = cyl(f, g)$  and the statement follows from proposition 2.4.1.

If  $X, Y$  and  $Z$  are simplicial objects, by diagram (2.12) of lemma 2.3.3 we deduce that  $sj_Z \in E$  if and only if  $I_{sZ} \in E$ . Then it follows from the constant case that this holds if and only if  $sf \in E$ .

Similarly,  $sj_Y$  if and only if  $I_{sY}$  is so, if and only if  $sg$  is so, by the constant case.  $\square$

**THEOREM 2.4.3.** *Let  $f : X \rightarrow Y$  be a morphism in  $\Delta^\circ \mathcal{D}$  and  $\epsilon : X \rightarrow X_{-1} \times \Delta$  an augmentation. Then*

**a)** *if  $sf$  is an equivalence then the simple of  $i_{X_{-1}} : X_{-1} \times \Delta \rightarrow Cyl(f, \epsilon)$  is so.*

**b)** *if  $s\epsilon$  is an equivalence then the simple of  $i_Y : Y \rightarrow Cyl(f, \epsilon)$  is so.*

*Proof.* Assume that  $\mathbf{s}f \in E$ . Then we deduce from 2.4.2 that  $\mathbf{s}j_{X_{-1}} : \mathbf{R}X_{-1} \rightarrow \mathbf{s}\widetilde{Cyl}(f, \epsilon)$  is in  $E$ .

On the other hand, by 1.7.14 we deduce that the following diagram commutes in  $\mathcal{D}$

$$\begin{array}{ccccc} \mathbf{s}(Cyl(f, \epsilon)) & \xrightarrow{\mathbf{s}\alpha} & \mathbf{s}(\widetilde{Cyl}(f, \epsilon)) & \xrightarrow{\mathbf{s}\beta} & \mathbf{s}(Cyl(f, \epsilon)) \\ \uparrow \mathbf{s}(i_{X_{-1}}) & & \uparrow \mathbf{s}(j_{X_{-1}}) & & \uparrow \mathbf{s}(i_{X_{-1}}) \\ \mathbf{R}X_{-1} & \xrightarrow{Id} & \mathbf{R}X_{-1} & \xrightarrow{Id} & \mathbf{R}X_{-1} \end{array},$$

where  $\mathbf{s}\beta \cdot \mathbf{s}\alpha = Id$ .

Therefore  $\mathbf{s}(i_{X_{-1}})$  is a retract of  $\mathbf{s}(j_{X_{-1}}) \in E$ , and by 2.2.11  $\mathbf{s}(i_{X_{-1}}) \in E$ .

To see b), one can argue in a similar way.  $\square$

The previous result remains valid using the symmetric notion of cylinder,  $Cyl'$ , introduced in 1.6.2.

**COROLLARY 2.4.4.** *Let  $f : X \rightarrow Y$  be a morphism in  $\Delta^\circ \mathcal{D}$  and  $\epsilon : X \rightarrow X_{-1} \times \Delta$  be an augmentation. Then*

*a) if  $\mathbf{s}f$  is an equivalence, the simple of  $i_{X_{-1}} : X_{-1} \times \Delta \rightarrow Cyl'(f, \epsilon)$  is so.*

*b) if  $\mathbf{s}\epsilon$  is an equivalence, the simple of  $i_Y : Y \rightarrow Cyl'(f, \epsilon)$  is so.*

*Proof.* Again, the statement follows from the previous proposition together with (SDC 8). Let us see b), since a) can be proved analogously.

If  $\mathbf{s}\epsilon \in E$  then  $\mathbf{s}\Upsilon\epsilon$  is also in  $E$ . By the previous proposition, the simple of  $\Upsilon Y \rightarrow Cyl(\Upsilon f, \Upsilon\epsilon)$  is an equivalence.

Therefore the simple of  $Y \rightarrow \Upsilon Cyl(\Upsilon f, \Upsilon\epsilon) = Cyl'(f, \epsilon)$  is in  $E$ , and we are done.  $\square$

**THEOREM 2.4.5.** *Let  $X_{-1} \times \Delta \xleftarrow{\epsilon} X \xrightarrow{f} Y$  be a diagram in  $\Delta^\circ \mathcal{D}$  such that there exists  $\epsilon' : Y \rightarrow X_{-1} \times \Delta$  with  $\epsilon'f = \epsilon$ . Then*

*$\mathbf{s}f \in E$  if and only if  $\mathbf{s}i_{X_{-1}} : \mathbf{R}X_{-1} \rightarrow \mathbf{s}Cyl(f, \epsilon)$  is in  $E$ .*

*Proof.* The “only if” part follows from 2.4.3. Assume that  $\mathbf{s}i_{X_{-1}} \in E$ . From the commutativity of the diagram of  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \epsilon \downarrow & & \downarrow \epsilon' \\ X_{-1} \times \Delta & \xrightarrow{Id} & X_{-1} \times \Delta \end{array}$$

we get by 1.5.10 the existence of a morphism  $H : Cyl(f, \epsilon) \rightarrow X_{-1} \times \Delta$  such that  $H \circ i_{X_{-1}} = Id_{X_{-1} \times \Delta}$  and  $H \circ i_Y = \epsilon'$ .

Since  $\mathbf{s}i_{X_{-1}} \in E$ , it follows from the 2 out of 3 property that  $\mathbf{s}H$  is an equivalence.

On the other hand, applying 1.6.8 to the diagram

$$\begin{array}{ccccc}
 & X_{-1} \times \Delta & \longrightarrow & 1 \times \Delta & \\
 & \swarrow Id & & \uparrow & \\
 X_{-1} \times \Delta & \xrightarrow{\epsilon} & X & \xrightarrow{Id} & X \\
 \uparrow \epsilon' & \swarrow f & & \swarrow f & \\
 Y & \xrightarrow{Id} & Y & & 
 \end{array}$$

we get an isomorphism between the simplicial object obtained as the image under  $Cyl$  of

$$1 \times \Delta \longleftarrow Cyl'(Id_X, \epsilon) \xrightarrow{F} Cyl'(Id_Y, \epsilon')$$

and the image under  $Cyl'$  of

$$X_{-1} \times \Delta \xleftarrow{H} Cyl(f, \epsilon) \xrightarrow{G} C(f) .$$

In other words,  $C(F) \simeq Cyl'(G, H)$ .

As  $\mathbf{s}(Id_X) = Id_{\mathbf{s}X} \in E$  then by 2.4.4 we deduce that the simple of the canonical inclusion  $j_{X_{-1}} : X_{-1} \times \Delta \rightarrow Cyl'(Id_X, \epsilon)$  is an equivalence. Similarly, the same holds for the simple of  $l_{X_{-1}} : X_{-1} \times \Delta \rightarrow Cyl'(Id_Y, \epsilon')$ .

Moreover, from the naturality of  $Cyl'$  we get that  $F \circ j_{X_{-1}} = l_{X_{-1}}$ , hence  $\mathbf{s}F \in E$ . Then, by the acyclicity axiom,  $\mathbf{s}C(F) \rightarrow 1$  is in  $E$ , consequently  $\mathbf{s}Cyl'(G, H) \rightarrow 1$  is also an equivalence.

But  $\mathbf{s}H \in E$ , and again it follows from 2.4.4 that the simple of  $C(f) \rightarrow Cyl(G, H)$  is in  $E$ . Therefore  $\mathbf{s}C(f)$  is acyclic, and from the acyclicity criterion we deduce that  $\mathbf{s}f \in E$ .  $\square$

**PROPOSITION 2.4.6.**

- i) If  $f, g$  are homotopic morphisms in  $\Delta^\circ \mathcal{D}$  (see 1.2.9) then  $\mathbf{s}(f) = \mathbf{s}(g)$  in  $Ho\mathcal{D}$ .
- ii) If  $\epsilon : X \rightarrow X_{-1} \times \Delta$  has a (lower or upper) extra degeneracy (see 1.2.10) then  $\mathbf{s}(\epsilon)$  is an equivalence.

*Proof.* Let  $\widetilde{Cyl}(X) = \widetilde{Cyl}(Id_X, Id_X)$  be the cubical cylinder object associated with  $X$ , given in 1.3.18. The morphisms  $f$  and  $g$  are homotopic in  $\Delta^\circ \mathcal{D}$ , so there

exists a homotopy  $H : \widetilde{Cyl}(X) \rightarrow X$  such that the following diagram commutes in  $\Delta^\circ \mathcal{D}$

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ & \searrow I & \nearrow H \\ & \widetilde{Cyl}(X) & \\ & \nearrow J & \searrow g \\ X & \xrightarrow{g} & X \end{array}$$

Then  $\mathbf{s}(f) = \mathbf{s}(H) \circ \mathbf{s}(I)$  and  $\mathbf{s}(g) = \mathbf{s}(H) \circ \mathbf{s}(J)$  in  $\mathcal{D}$ , so it is enough to check the equality  $\mathbf{s}(I) = \mathbf{s}(J)$  in  $Ho\mathcal{D}$ .

If  $cyl(\mathbf{s}X) = cyl(Id_{\mathbf{s}X}, Id_{\mathbf{s}X})$ , by 2.3.3, it suffices to see that the inclusions  $I_{\mathbf{s}X}$  and  $J_{\mathbf{s}X} : R\mathbf{s}X \rightarrow cyl(\mathbf{s}X)$  coincide in  $Ho\mathcal{D}$ . Note that both morphisms  $I_{\mathbf{s}X}$ ,  $J_{\mathbf{s}X}$  are equivalences, because of 2.4.1.

On the other hand, it follows from 2.2.9 the existence of  $\rho : cyl(\mathbf{s}(X)) \rightarrow R\mathbf{s}(X)$  such that  $\rho \circ I_{R\mathbf{s}(X)} = \rho \circ J_{R\mathbf{s}(X)} = Id$ .

Hence, in  $Ho\mathcal{D}$ ,  $\rho$  is an isomorphism such that  $I_{R\mathbf{s}(X)} = J_{R\mathbf{s}(X)} = \rho^{-1}$ , that finish the proof. *ii)* follows from *i)*, having into account 1.2.12.  $\square$

**COROLLARY 2.4.7.** *If  $D$  is the object  $X_{-1} \times \Delta \leftarrow X \rightarrow Y$  of  $\Omega(\mathcal{D})$ , the following diagram commutes in  $Ho\mathcal{D}$*

$$\begin{array}{ccc} \mathbf{s}(X) & \xrightarrow{\mathbf{s}(f)} & \mathbf{s}(Y) \\ \mathbf{s}(\epsilon) \downarrow & & \downarrow \mathbf{s}(i_Y) \\ \mathbf{s}(X_{-1} \times \Delta) & \xrightarrow{\mathbf{s}(i_{X_{-1}})} & \mathbf{s}(Cyl(D)). \end{array}$$

*Proof.* In the first chapter we proved that  $i_Y \circ f$  is homotopic to  $i_{X_{-1}} \circ \epsilon$  (see 1.5.9), therefore the statement is a consequence of the above proposition.  $\square$

**COROLLARY 2.4.8.** *If  $f : X \rightarrow Y$  is a morphism between simplicial objects then the composition  $\mathbf{s}(X) \xrightarrow{\mathbf{s}(f)} \mathbf{s}(Y) \rightarrow \mathbf{s}(Cf)$  is trivial in  $Ho\mathcal{D}$ , that is, it factors through the final object 1.*

*Proof.* By the previous proposition we have the following commutative diagram in  $Ho\mathcal{D}$

$$\begin{array}{ccc} \mathbf{s}(X) & \xrightarrow{\mathbf{s}(f)} & \mathbf{s}(Y) \\ \downarrow & & \downarrow \mathbf{s}(i_Y) \\ R1 & \longrightarrow & \mathbf{s}(C(f)). \end{array}$$

The result follows from the equivalence existing between  $R1$  and  $1$  (by (SDC 5)), so they are isomorphic in  $Ho\mathcal{D}$ .  $\square$

**COROLLARY 2.4.9.** *Given an object  $Z \xleftarrow{g} X \xrightarrow{f} Y$  in  $\square_1^{\circ} \mathcal{D}$ , then the following diagram commutes in  $Ho\mathcal{D}$*

$$\begin{array}{ccc} RX & \xrightarrow{Rf} & RY \\ Rg \downarrow & & \downarrow I_Y \\ RZ & \xrightarrow{I_Z} & cyl(f, g) . \end{array}$$

Moreover, if  $cyl(X) = cyl(Id_X, Id_X)$ , there exists  $H : cyl(X) \rightarrow cyl(f, g)$  such that  $H \circ I_X = I_Y \circ Rf$  and  $H \circ J_X = I_Z \circ Rg$ , where  $I, J$  are the inclusions of  $X$  into  $cyl(X)$ .

In particular, the composition  $RX \xrightarrow{Rf} RY \xrightarrow{I_Y} c(f)$  is trivial in  $Ho\mathcal{D}$ .

*Proof.* Having in mind the commutativity up to homotopy of the diagram of simplicial objects

$$\begin{array}{ccc} X \times \Delta & \xrightarrow{f \times \Delta} & Y \times \Delta \\ \downarrow & & \downarrow i_Y \\ Z \times \Delta & \xrightarrow{i_Z} & Cyl(f \times \Delta, g \times \Delta) , \end{array}$$

there exists  $L : \widetilde{Cyl}(X \times \Delta) \rightarrow Cyl(f \times \Delta, g \times \Delta)$  such that  $L \circ i_X = i_Y \circ f \times \Delta$  and  $L \circ j_X = i_Z \circ g \times \Delta$ . In addition,  $\mathbf{s}\widetilde{Cyl}(X \times \Delta)$  is equal to  $cyl(X)$ , therefore it is enough to take  $H = \mathbf{s}L$ .  $\square$

**REMARK 2.4.10.** Consider the commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccccc} X & \xleftarrow{g} & Y & \xrightarrow{f} & Z \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ X' & \xleftarrow{g'} & Y' & \xrightarrow{f'} & Z' . \end{array}$$

From the functoriality of  $cyl$  we obtain  $\delta : cyl(f, g) \rightarrow cyl(f', g')$  such that in the diagram

$$\begin{array}{ccccc} & & RY & \xrightarrow{Rf} & RZ \\ & Rg \swarrow & \downarrow I_X & \swarrow I_Z & \downarrow \gamma \\ RX & \xrightarrow{\quad} & cyl(f, g) & & \\ \alpha \downarrow & & \downarrow \beta & & \\ & Rg' \swarrow & RY' & \xrightarrow{\quad} & RZ' , \\ & & \downarrow I_{X'} & \swarrow I_{Z'} & \\ RX' & \xrightarrow{I_{X'}} & cyl(f', g') & & \end{array}$$



all the faces commute in  $\mathcal{D}$ , except the lower and upper ones, that commute in  $Ho\mathcal{D}$ .

To finish this section, we give the following result just for completeness, because we will not use it in these notes. It is a consequence of proposition 2.4.1.

**PROPOSITION 2.4.11.** *If  $f, g$  are morphisms in  $\mathcal{D}$  such that  $f \sqcup g$  is an equivalence, then  $f$  and  $g$  are so.*

*Proof.* Let  $f : A \rightarrow B$  and  $g : A' \rightarrow B'$  be morphisms such that  $f \sqcup g \in E$ , and let us prove that  $f \in E$ . We will check that the canonical inclusion  $R1 \rightarrow c(f)$  is in  $E$  (that is equivalent to check that  $c(f) \rightarrow 1$  is in  $E$ ).

Consider the trivial morphisms  $\rho_A : A \rightarrow 1$  and  $\rho_{A'} : A' \rightarrow 1$ , as well as the morphism  $\rho = \rho_A \sqcup \rho_{A'} : A \sqcup A' \rightarrow 1 \sqcup 1$ . Since  $f \sqcup g \in E$ , we deduce from 2.4.1 that the inclusion

$$I : R(1 \sqcup 1) \rightarrow cyl(f \sqcup g, \rho)$$

is an equivalence. If we denote by  $I_f : R1 \rightarrow c(f)$  and  $I_g : R1 \rightarrow c(g)$  the canonical inclusions, by 2.2.8 we get an equivalence  $\sigma_{cyl}$  such that the diagram

$$\begin{array}{ccc} R1 \sqcup R1 & \xrightarrow{I_f \sqcup I_g} & c(f) \sqcup c(g) \\ \sigma_R \downarrow & & \downarrow \sigma_{cyl} \\ R(1 \sqcup 1) & \xrightarrow{I} & cyl(f \sqcup g, \rho) \end{array}$$

commutes. Moreover, from (SDC 3) we deduce that  $\sigma_R \in E$ , and therefore  $I_f \sqcup I_g$  is an equivalence. Finally, it is enough to see that  $I_f$  is a retract of  $I_f \sqcup I_g$ , in such a case the proof would be concluded by 2.2.11.

The “zero” morphism  $\alpha : c(g) \rightarrow c(f)$  is defined as follows. The morphism  $C(g \times \Delta) \rightarrow 1 \times \Delta$  gives rise to  $c(g) \rightarrow R1$ , and by composing with  $I_f$  we get the desired morphism  $\alpha$ . Moreover,  $\alpha \circ I_g = I_f$  since at the simplicial level  $1 \times \Delta \rightarrow C(g \times \Delta) \rightarrow 1 \times \Delta$  is the identity.

Then, we obtain the commutative diagram

$$\begin{array}{ccccc} c(f) & \longrightarrow & c(f) \sqcup c(g) & \xrightarrow{Id \sqcup \alpha} & c(f) \\ \uparrow I_f & & \uparrow I_f \sqcup I_g & & \uparrow I_f \\ R1 & \longrightarrow & R1 \sqcup R1 & \xrightarrow{Id \sqcup Id} & R1 \end{array}$$

where the horizontal compositions are the identity, and it follows that  $I_f$  is in fact a retract of  $I_f \sqcup I_g$ .  $\square$

## 2.5 Functors of simplicial descent categories

The aim of this section is to state and prove the “transfer lemma”, that will allow us to transfer the simplicial descent structure from a fixed simplicial descent category to a new category through a suitable functor.

**DEFINITION 2.5.1.** Let  $(\mathcal{D}, \mathbf{s}, E, \mu, \lambda)$  and  $(\mathcal{D}', E', \mathbf{s}', \mu', \lambda')$  be simplicial descent categories. We say that a functor  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  is a *functor of simplicial descent categories* if

(FD 0)  $\psi$  preserves equivalences, that is,  $\psi(E) \subseteq E'$ .

(FD 1)  $\psi$  is a quasi-strict monoidal functor with respect to the coproduct (see 2.1.4).

(FD 2) Consider the diagram

$$\begin{array}{ccc} \Delta^\circ \mathcal{D} & \xrightarrow{\Delta^\circ \psi} & \Delta^\circ \mathcal{D}' \\ \mathbf{s} \downarrow & & \downarrow \mathbf{s}' \\ \mathcal{D} & \xrightarrow{\psi} & \mathcal{D}' \xrightarrow{\gamma} Ho\mathcal{D}' \end{array} \quad (2.16)$$

There exists a natural isomorphism of functors  $\Theta : \gamma \circ \psi \circ \mathbf{s} \rightarrow \gamma \circ \mathbf{s}' \circ \Delta^\circ \psi$  in such a way that  $\Theta$  comes from a functorial “zig-zag” with values in  $\mathcal{D}'$ . Moreover,  $\Theta$  must be compatible with  $\lambda, \lambda'$  and with  $\mu, \mu'$ .

More concretely, there exists functors  $A^0, \dots, A^r : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}'$  such that  $A^0 = \psi \circ \mathbf{s}$  and  $A^r = \mathbf{s}' \circ \Delta^\circ \psi$ , and they are related by the natural transformations

$$\psi \circ \mathbf{s} = A^0 \xrightarrow{\Theta^0} A^1 \xrightarrow{\Theta^1} \dots \xrightarrow{\Theta^{r-1}} A^r = \mathbf{s}' \circ \Delta^\circ \psi \quad (2.17)$$

where either  $\Theta^i : A^i \rightarrow A^{i+1}$  or  $\Theta^i : A^{i+1} \rightarrow A^i$ , and such that  $\Theta_X^i \in E'$  for all  $X$  in  $\Delta^\circ \mathcal{D}$  and for all  $i$ . The natural transformation  $\Theta$  must be the image under  $\gamma$  of the zig-zag (2.17).

**(2.5.2)** Let us describe more specifically the compatibility condition that is mentioned in (FD 2). We will denote also by  $\psi$  the induced morphisms  $\Delta^\circ \psi : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}'$  and  $\Delta^\circ \Delta^\circ \psi : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}'$ .

I. Given an object  $X$  in  $\mathcal{D}$ , the following diagram must commute in  $Ho\mathcal{D}'$

$$\begin{array}{ccc} \psi(\mathbf{s}(X \times \Delta)) & \xrightarrow{\psi(\lambda_X)} & \psi(X) \\ \Theta_{X \times \Delta} \downarrow & \nearrow \lambda'_{\psi(X)} & \\ \mathbf{s}'(\psi(X) \times \Delta) & & \end{array} \quad (2.18)$$

II. If  $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$ , then  $\psi(DZ) = D(\psi(Z))$  and  $\Theta_{DZ} : \gamma\psi(s(DZ)) \rightarrow \gamma s'D(\psi(Z))$ . The following diagram must be commutative in  $Ho\mathcal{D}'$

$$\begin{array}{ccc} s'D(\psi Z) & \xrightarrow{\mu'_{\psi(Z)}} & s'\Delta^\circ s'(\psi Z) \\ \Theta_{DZ} \uparrow & & \uparrow (s'\Delta^\circ \Theta \circ \Theta)_Z \\ \psi s(DZ) & \xrightarrow{\psi(\mu_Z)} & \psi s(\Delta^\circ s Z), \end{array} \quad (2.19)$$

where the natural transformation  $s'\Delta^\circ \Theta \circ \Theta : \psi \circ s \circ \Delta^\circ s \rightarrow s' \circ \Delta^\circ s' \circ \psi$  is defined in  $Ho\mathcal{D}'$  and is induced in a natural way by  $\Theta$ .

Concretely, on one hand we have that

$$\Theta_{\Delta^\circ s Z} : \psi s(\Delta^\circ s Z) \longrightarrow s' \psi(\Delta^\circ s Z) .$$

On the other hand, fixing as  $n$  the first index of  $Z$  we get  $Z_{n,\cdot} \in \Delta^\circ \mathcal{D}$ , and the evaluation of (2.17) in  $Z_{n,\cdot}$  is the natural sequence of morphisms in  $\mathcal{D}'$

$$\psi(s Z_{n,\cdot}) \xrightarrow{\Theta_{Z_{n,\cdot}}^0} A^1(Z_{n,\cdot}) \xrightarrow{\Theta_{Z_{n,\cdot}}^1} \dots \xrightarrow{\Theta_{Z_{n,\cdot}}^{r-1}} s'(\psi Z_{n,\cdot}) .$$

Therefore, we obtain the sequence in  $\Delta^\circ \mathcal{D}'$

$$\psi(\Delta^\circ s Z) \xrightarrow{\Delta^\circ \Theta_Z^0} \Delta^\circ A^1 Z \xrightarrow{\Delta^\circ \Theta_Z^1} \dots \xrightarrow{\Delta^\circ \Theta_Z^{r-1}} \Delta^\circ s'(\psi Z) .$$

and, by the exactness axiom, the result of applying  $s'$  is the sequence of equivalences in  $\mathcal{D}'$

$$s' \psi(\Delta^\circ s Z) \longrightarrow s' \Delta^\circ A^1 Z \longrightarrow \dots \longrightarrow s' \Delta^\circ s'(\psi Z) .$$

that gives rise to the morphism in  $Ho\mathcal{D}'$

$$(s'\Delta^\circ \Theta)_Z : s'\Delta^\circ(\psi s)Z \longrightarrow s'\Delta^\circ s'(\psi Z)$$

Then,  $(\Theta \circ s'\Delta^\circ \Theta)_Z$  is the composition

$$\psi s(\Delta^\circ s Z) \xrightarrow{\Theta_{\Delta^\circ s Z}} s' \psi(\Delta^\circ s Z) \xrightarrow{s'(\Delta^\circ \Theta)_Z} s' \circ \Delta^\circ s'(\psi Z) .$$

**REMARK 2.5.3** (Case  $\Theta = Id$ ). Assume that (2.16) is commutative, that is,  $\Theta = Id : \psi \circ s \rightarrow s' \circ \Delta^\circ \psi$  as functors  $\Delta^\circ \mathcal{D} \rightarrow \mathcal{D}'$ . In this case the compatibility condition between  $\lambda$ ,  $\lambda'$  and  $\mu$ ,  $\mu'$  means that the following equalities hold in  $Ho\mathcal{D}'$

$$\psi(\lambda(X)) = \lambda'(\psi(X)) \quad \forall X \in \mathcal{D} \quad ; \quad \psi(\mu(Z)) = \mu'(\psi(Z)) \quad \forall Z \in \Delta^\circ \Delta^\circ \mathcal{D} .$$

**EXAMPLE 2.5.4.** If  $F : \mathcal{D} \rightarrow \mathcal{D}'$  is a functor of simplicial descent categories and  $I$  is a small category, then  $F_* : I\mathcal{D} \rightarrow I\mathcal{D}'$  is also a functor of simplicial descent categories, with the descent structures introduced in 2.1.15.

In addition, if  $\mathcal{D}$  is a simplicial descent category and  $x$  is an object in  $\mathcal{D}$ , the “evaluation at”  $x$  functor,  $ev_x : I\mathcal{D} \rightarrow \mathcal{D}$  with  $ev_x(f) = f(x)$ , is a functor of simplicial descent categories.

**REMARK 2.5.5.** In the following lemma we will prove that the composition of two functors of simplicial descent categories is again a functor in this way. Hence, we have the category (in a convenient universe)  $\mathfrak{DesSimp}$  of simplicial descent categories together with the functors of simplicial descent categories.

**LEMMA 2.5.6.** *The composition of two functors of simplicial descent categories is again a functor of simplicial descent categories.*

*Proof.* Let  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  and  $\psi' : \mathcal{D}' \rightarrow \mathcal{D}''$  be functors of simplicial descent categories. We shall study the commutativity of the diagrams

$$\begin{array}{ccccc} \Delta^\circ \mathcal{D} & \xrightarrow{\Delta^\circ \psi} & \Delta^\circ \mathcal{D}' & \xrightarrow{\Delta^\circ \psi'} & \Delta^\circ \mathcal{D}'' \\ \downarrow \mathbf{s} & & \downarrow \mathbf{s}' & & \downarrow \mathbf{s}'' \\ \mathcal{D} & \xrightarrow{\psi} & \mathcal{D}' & \xrightarrow{\psi'} & \mathcal{D}'' \end{array} .$$

Assume that the zig-zag  $\Theta$  and  $\Phi$  associated with  $\psi$  and  $\psi'$  are given respectively by

$$\begin{aligned} \psi \circ \mathbf{s} &= A^0 \xrightarrow{\Theta^0} A^1 \xrightarrow{\Theta^1} \dots \xrightarrow{\Theta^{r-1}} A^r = \mathbf{s}' \circ \Delta^\circ \psi \\ \psi' \circ \mathbf{s}' &= B^0 \xrightarrow{\Phi^0} B^1 \xrightarrow{\Phi^1} \dots \xrightarrow{\Phi^{s-1}} B^s = \mathbf{s}'' \circ \Delta^\circ \psi' \end{aligned}$$

Then  $\Psi = (\Phi \circ \Delta^\circ \psi) \circ (\psi' \circ \Theta)$  is the zig-zag relating  $\psi' \circ \psi \circ \mathbf{s}$  to  $\mathbf{s}'' \circ \Delta^\circ (\psi' \circ \psi)$ , whose value at  $X \in \Delta^\circ \mathcal{D}$  is

$$\psi' \psi \mathbf{s} X \xrightarrow{\psi' \Theta_X^0} \dots \xrightarrow{\psi' \Theta_X^{r-1}} \psi' \mathbf{s}' \circ \Delta^\circ \psi X \xrightarrow{\Phi_{\Delta^\circ \psi X}^0} \dots \xrightarrow{\Phi_{\Delta^\circ \psi X}^{s-1}} \mathbf{s}'' \Delta^\circ (\psi' \psi) X$$

The compatibility of  $\Psi$  with  $\lambda$  and  $\lambda''$  follows from the one of  $\Theta$  and  $\Phi$  with  $\lambda$ ,  $\lambda'$  and  $\lambda''$ . On the other hand, if  $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$ , the square (2.19) can be drawn in this case as

$$\begin{array}{ccc} \mathbf{s}'' \mathbf{D}(\psi' \psi Z) & \xrightarrow{\mu''_{\psi' \psi Z}} & \mathbf{s}'' \Delta^\circ \mathbf{s}''(\psi' \psi Z) \\ \uparrow \Psi_{\mathbf{D}Z} & & \uparrow (\mathbf{s}'' \Delta^\circ \Psi)_Z \\ \psi' \psi \mathbf{s}(\mathbf{D}Z) & \xrightarrow{\psi' \psi \mu_Z} & \psi' \psi \mathbf{s}(\Delta^\circ \mathbf{s} Z) \end{array} \quad (2.20)$$

It is enough to check that  $(s'' \Delta^\circ \Psi)_{Z \circ \Psi_{\Delta^\circ sZ}}$  agrees with

$$\begin{aligned} \psi' \psi s(\Delta^\circ sZ) &\xrightarrow{\psi' \Theta_{\Delta^\circ sZ}} \psi' s' \psi(\Delta^\circ sZ) \xrightarrow{\psi'(s' \Delta^\circ \Theta)_Z} \psi' s' \Delta^\circ s'(\psi Z) \\ \psi' s' \Delta^\circ s'(\psi Z) &\xrightarrow{\Phi_{\Delta^\circ s'(\psi Z)}} s'' \psi' s'(\psi Z) \xrightarrow{(s'' \Delta^\circ \Phi)_{\psi Z}} s'' \Delta^\circ s''(\psi' \psi Z) \end{aligned} \quad (2.21)$$

since in this case diagram (2.20) is the composition of the following commutative squares

$$\begin{array}{ccc} \psi' s' D(\psi Z) & \xrightarrow{\psi' \mu'_{\psi Z}} & \psi' s' \Delta^\circ s'(\psi Z) & & s'' D(\psi' \psi Z) & \xrightarrow{\mu''_{\psi' \psi Z}} & s'' \Delta^\circ s''(\psi' \psi Z) \\ \uparrow \psi' \Theta_{DZ} & & \uparrow \psi'(s' \Delta^\circ \Theta)_Z & & \uparrow \Phi_{\psi DZ} & & \uparrow (s'' \Delta^\circ \Phi)_{\psi Z} \\ \psi' \psi s(DZ) & \xrightarrow{\psi' \psi(\mu_Z)} & \psi' \psi s(\Delta^\circ sZ) & & \psi' s' D(\psi Z) & \xrightarrow{\psi' \mu'_{\psi(Z)}} & \psi' s' \Delta^\circ s'(\psi Z) \\ & & \uparrow \psi' \Theta_{\Delta^\circ sZ} & & & & \uparrow \Phi_{\Delta^\circ s'(\psi Z)} \end{array}$$

By definition  $(s'' \Delta^\circ \Psi)_{Z \circ \Psi_{\Delta^\circ sZ}}$  is the zig-zag

$$\begin{aligned} \psi' \psi s(\Delta^\circ sZ) &\xrightarrow{\psi' \Theta_{\Delta^\circ sZ}} \psi' s' \psi(\Delta^\circ sZ) \xrightarrow{\Phi_{\psi(\Delta^\circ sZ)}} s'' \psi' \psi(\Delta^\circ sZ) \\ s'' \psi' \psi(\Delta^\circ sZ) &\xrightarrow{(s'' \psi' \Delta^\circ \Theta)_Z} s'' \psi' s' \psi(\Delta^\circ sZ) \xrightarrow{(s'' \Delta^\circ \Phi)_Z} s'' \Delta^\circ s''(\psi' \psi Z) \end{aligned}$$

and to see that it agrees with (2.21), it suffices to prove that the following diagram is commutative in  $Ho\mathcal{D}''$

$$\begin{array}{ccc} \psi' s' \psi(\Delta^\circ sZ) & \xrightarrow{\Phi_{\psi(\Delta^\circ sZ)}} & s'' \psi' \psi(\Delta^\circ sZ) \\ \downarrow \psi'(s' \Delta^\circ \Theta)_Z & & \downarrow (s'' \psi' \Delta^\circ \Theta)_Z \\ \psi' s' \Delta^\circ s'(\psi Z) & \xrightarrow{\Phi_{\Delta^\circ s'(\psi Z)}} & s'' \psi' s'(\psi Z) \end{array}$$

Expanding this diagram we get

$$\begin{array}{ccc} \psi' s' \psi(\Delta^\circ sZ) & \xrightarrow{\Phi_{\psi(\Delta^\circ sZ)}^0} B^1(\psi(\Delta^\circ sZ)) \cdots B^{s-1}(\psi(\Delta^\circ sZ)) & \xrightarrow{\Phi_{\psi(\Delta^\circ sZ)}^{s-1}} s'' \psi' \psi(\Delta^\circ sZ) \\ \downarrow \psi'(s' \Delta^\circ \Theta^0)_Z & & \downarrow (s'' \psi' \Delta^\circ \Theta^0)_Z \\ \psi' s'(\Delta^\circ A^1 Z) & & s'' \psi'(\Delta^\circ A^1 Z) \\ \vdots & & \vdots \\ \downarrow \psi'(s' \Delta^\circ \Theta^{r-1})_Z & & \downarrow (s'' \psi' \Delta^\circ \Theta^{r-1})_Z \\ \psi' s' \Delta^\circ s'(\psi Z) & \xrightarrow[\Phi_{\Delta^\circ s'(\psi Z)}^0]{} B^1(\Delta^\circ s'(\psi Z)) \cdots B^{s-1}(\Delta^\circ s'(\psi Z)) & \xrightarrow[\Phi_{\Delta^\circ s'(\psi Z)}^{s-1}]{} s'' \psi' s' \psi(\Delta^\circ sZ) \end{array}$$

Since  $\Phi^j$  is a natural transformation relating  $B^j$  and  $B^{j+1}$ , we deduce that the two upper rows of the above square can be completed to the following diagram, where each square commutes

$$\begin{array}{ccccccc}
\psi' s' \psi(\Delta^\circ s Z) & \xrightarrow{\Phi_{\psi(\Delta^\circ s Z)}^0} & B^1(\psi(\Delta^\circ s Z)) & \cdots & B^{s-1}(\psi(\Delta^\circ s Z)) & \xrightarrow{\Phi_{\psi(\Delta^\circ s Z)}^{s-1}} & s'' \psi' \psi(\Delta^\circ s Z) \\
\left| \psi'(s' \Delta^\circ \Theta^0)_Z \right. & & \left| B^1(\Delta^\circ \Theta^0)_Z \right. & & \left| B^{s-1}(\Delta^\circ \Theta^0)_Z \right. & & \left| (s'' \psi' \Delta^\circ \Theta^0)_Z \right. \\
\psi' s'(\Delta^\circ A^1 Z) & \xrightarrow[\Phi_{\psi(\Delta^\circ A^1 Z)}^0]{} & B^1(\Delta^\circ A^1 Z) & \cdots & B^{s-1}(\Delta^\circ A^1 Z) & \xrightarrow[\Phi_{\psi(\Delta^\circ A^1 Z)}^{s-1}]{} & s'' \psi'(\Delta^\circ A^1 Z)
\end{array}$$

and iterating this procedure we get that the required diagram commutes.  $\square$

Next we introduce the “transfer lemma”. To this end we need the following remark about the commutativity between the simplicial cylinder functor and the functor  $\Delta^\circ \psi : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}'$  induced by a quasi-strict monoidal functor  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$ .

**REMARK 2.5.7.** If  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  satisfies (FD 1), that is, it is quasi-strict monoidal, then the diagram

$$\begin{array}{ccc}
\Omega(\mathcal{D}) & \xrightarrow{\Delta^\circ \psi} & \Omega(\mathcal{D}') \\
Cyl \downarrow & & \downarrow Cyl \\
\Delta^\circ \mathcal{D} & \xrightarrow{\Delta^\circ \psi} & \Delta^\circ \mathcal{D}'
\end{array}$$

commutes up to (degreewise) equivalence.

More concretely, there exists  $\tau : Cyl \Delta^\circ \psi \rightarrow \Delta^\circ \psi Cyl$  such that  $(\tau_D)_n \in E'$   $\forall D \in \Omega(\mathcal{D}), \forall n \geq 0$ .

Indeed, if  $D \equiv X_{-1} \times \Delta \xleftarrow{\varepsilon} X \xrightarrow{f} Y$  then

$$\Delta^\circ \psi(\mathcal{D}) \equiv \psi(X_{-1}) \times \Delta \xleftarrow{\Delta^\circ \psi(\varepsilon)} \Delta^\circ \psi(X) \xrightarrow{\Delta^\circ \psi(f)} \Delta^\circ \psi(Y).$$

The morphisms  $(\tau_D)_n = \sigma_\psi : \psi(Y_n) \sqcup \psi(X_{n-1}) \sqcup \cdots \sqcup \psi(X_{-1}) \rightarrow \psi(Y_n \sqcup X_{n-1} \sqcup \cdots \sqcup X_{-1})$  are equivalences, and  $\tau_D = \{(\tau_D)_n\}_n$  is a morphism in  $\Delta^\circ \mathcal{D}'$ , due to the universal property of the coproduct.

Note also that by definition of  $i_{X_{-1}} : X_{-1} \times \Delta \rightarrow Cyl(D)$  and  $i_{\psi(X_{-1})} : \psi(X_{-1}) \times \Delta \rightarrow Cyl(\Delta^\circ \psi(D))$ , the diagram

$$\begin{array}{ccc}
\psi(X_{-1}) \times \Delta & \xrightarrow{\Delta^\circ \psi(i_{X_{-1}})} & \Delta^\circ \psi(Cyl(D)) \\
\searrow i_{\psi(X_{-1})} & & \uparrow \tau_D \\
& & Cyl(\Delta^\circ \psi(D))
\end{array} \tag{2.22}$$

commutes.

**THEOREM 2.5.8** (Transfer lemma).

Consider the data  $(\mathcal{D}, \mathbf{s}, \mu, \lambda)$  satisfying

(SDC 1)  $\mathcal{D}$  is a category with finite coproducts and final object 1.

(SDC 3)'  $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  is a functor.

(SDC 4)'  $\mu : \mathbf{sD} \rightarrow \mathbf{s}(\Delta^\circ \mathbf{s})$  is a natural transformation.

(SDC 5)'  $\lambda : \mathbf{s}(- \times \Delta) \rightarrow Id_{\mathcal{D}}$  is a natural transformation compatible with  $\mu$ , that is, the equalities in (2.1) hold.

Assume that  $(\mathcal{D}', \mathbf{E}', \mathbf{s}', \mu', \lambda')$  is a simplicial descent category and  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  a functor such that (FD 1) and (FD 2) hold.

Then, taking  $\mathbf{E} = \{f \mid \psi(f) \in \mathbf{E}'\}$ ,  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  is a simplicial descent category.

In addition,  $\psi : \mathcal{D} \rightarrow \mathcal{D}'$  is a functor of simplicial descent categories.

*Proof.* We must see that the data  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  satisfy the axioms of simplicial descent category.

For clarity, assume that the functorial zig-zag (2.17) in  $\mathcal{D}'$  associated with  $\Theta : \gamma \circ \psi \circ \mathbf{s} \rightarrow \gamma \circ \mathbf{s}' \circ \Delta^\circ \psi$  is

$$\psi \circ \mathbf{s} \xleftarrow{\Theta^0} A \xrightarrow{\Theta^1} \mathbf{s}' \circ \Delta^\circ \psi .$$

(SDC 2) By definition,  $\mathbf{E}$  is the class consisting of those morphisms that are mapped by the composition

$$\mathcal{D} \xrightarrow{\psi} \mathcal{D}' \rightarrow Ho\mathcal{D}'$$

into isomorphisms. Hence  $\mathbf{E}$  is saturated. Let us check that  $\mathbf{E}$  is stable under coproducts.

Let  $f_j : X_j \rightarrow Y_j$  be morphisms in  $\mathcal{D}$  for  $j = 1, 2$  such that  $\psi(f_j) \in \mathbf{E}'$ . Since

$$\begin{array}{ccc} \psi(X_1 \sqcup X_2) & \xrightarrow{\psi(f_1 \sqcup f_2)} & \psi(Y_1 \sqcup Y_2) \\ \sigma_\psi \uparrow & & \uparrow \sigma_\psi \\ \psi(X_1) \sqcup \psi(X_2) & \xrightarrow{\psi(f_1) \sqcup \psi(f_2)} & \psi(Y_1) \sqcup \psi(Y_2) \end{array}$$

commutes and  $\sigma_\psi \in \mathbf{E}'$ , it follows from the 2 out of 3 property that  $\psi(f_1 \sqcup f_2) \in \mathbf{E}'$ , therefore  $f_1 \sqcup f_2 \in \mathbf{E}$ .

(SDC 3) If  $X, Y \in \Delta^\circ \mathcal{D}$ , we must see that  $\psi(\sigma_{\mathbf{s}}) : \psi(\mathbf{s}(X) \sqcup \mathbf{s}(Y)) \rightarrow \psi(\mathbf{s}(X \sqcup Y))$

is in  $E'$ .

Consider the diagram

$$\begin{array}{ccc}
\psi(\mathbf{s}X \sqcup \mathbf{s}Y) & \xrightarrow{\psi(\sigma_{\mathbf{s}})} & \psi\mathbf{s}(X \sqcup Y) \\
\uparrow \sigma_{\psi} & & \uparrow Id \\
\psi\mathbf{s}X \sqcup \psi\mathbf{s}Y & \xrightarrow{\sigma_{\psi\mathbf{s}}} & \psi\mathbf{s}(X \sqcup Y) \\
\uparrow \Theta_X^0 \sqcup \Theta_Y^0 & & \uparrow \Theta_{X \sqcup Y}^0 \\
A(X) \sqcup A(Y) & \xrightarrow{\sigma_A} & A(X \sqcup Y) \\
\downarrow \Theta_X^1 \sqcup \Theta_Y^1 & & \downarrow \Theta_{X \sqcup Y}^1 \\
\mathbf{s}'(\Delta^\circ \psi X) \sqcup \mathbf{s}'(\Delta^\circ \psi Y) & \xrightarrow{\sigma_{\mathbf{s}'\Delta^\circ \psi}} & \mathbf{s}'\Delta^\circ \psi(X \sqcup Y) \\
\downarrow \sigma_{\mathbf{s}'} & & \downarrow Id \\
\mathbf{s}'(\Delta^\circ \psi X \sqcup \Delta^\circ \psi Y) & \xrightarrow{\mathbf{s}'(\sigma_{\Delta^\circ \psi})} & \mathbf{s}'\Delta^\circ \psi(X \sqcup Y) .
\end{array}$$

The top and bottom squares commute in  $\mathcal{D}'$  (because of the universal property of the coproduct and the definition of  $\sigma$ , (2.1.4)). The two central squares commute by the same reason, since every natural transformation is monoidal with respect to the coproduct.

On the other and, the morphism  $\sigma_{\Delta^\circ \psi}$  is  $\sigma_{\psi} : \psi(X_n) \sqcup \psi(Y_n) \rightarrow \psi(X_n \sqcup Y_n)$  in degree  $n$ , that is in  $E'$  for all  $n$ . Therefore, from the exactness of  $\mathbf{s}'$  we deduce that  $\mathbf{s}'(\sigma_{\Delta^\circ \psi}) \in E'$ , and by the 2 out of 3 property we get that  $\sigma_{\mathbf{s}'\Delta^\circ \psi} \in E'$ . In addition  $\Theta_X^i \sqcup \Theta_Y^i, \Theta_{X \sqcup Y}^i \in E'$  for  $i = 0, 1$ , and hence  $\sigma_{\psi\mathbf{s}} \in E'$ . Consequently  $\psi(\sigma_{\mathbf{s}}) \in E'$ .

(SDC 4) Let  $Z \in \Delta^\circ \Delta^\circ \mathcal{D}$ . The square (2.19) commutes in  $Ho\mathcal{D}'$  and by definition the horizontal morphisms are isomorphisms, as well as  $\mu'_{\psi(Z)}$ .

Hence  $\psi(\mu_Z)$  is an isomorphism in  $Ho\mathcal{D}'$ , so it follows that  $\psi(\mu_Z) \in E'$ , and  $\mu_Z \in E$ .

(SDC 5) Analogously, given  $X \in \mathcal{D}$ , we deduce from the commutativity of (2.18) that  $\psi(\lambda_X) \in E'$ , so  $\lambda_X \in E$ .

(SDC 6) Let  $f : X \rightarrow Y$  be a morphism in  $\Delta^\circ \mathcal{D}$  with  $f_n \in E \ \forall n$ . Then  $[\Delta^\circ \psi(f)]_n = \psi(f_n) \in E' \ \forall n$ , and consequently  $\mathbf{s}'(\Delta^\circ \psi(f)) \in E'$ .



It follows from the naturality of  $\Theta$  that  $\mathbf{s}'(\Delta^\circ\psi(f)) \circ \Theta_X = \Theta_{Y \circ \psi}(\mathbf{s}(f))$  in  $Ho\mathcal{D}'$ , and hence  $\psi(\mathbf{s}(f)) \in E'$ , therefore  $\mathbf{s}(f) \in E$ .

(SDC 7) Given a morphism  $f : X \rightarrow Y$  in  $\Delta^\circ\mathcal{D}$  we have to prove that  $\psi\mathbf{s}f \in E'$  if and only if  $\psi\mathbf{s}(Cf) \rightarrow \psi(1)$  is so. By (FD 2), we have that  $\psi\mathbf{s}f \in E'$  if and only if  $\mathbf{s}'(\Delta^\circ\psi f) \in E'$ . Let  $\tau : Cyl \Delta^\circ\psi \rightarrow \Delta^\circ\psi Cyl$  be the morphism defined as in 2.5.7.

If  $D \equiv 1 \times \Delta \xleftarrow{\rho} X \xrightarrow{f} Y$ , applying  $\Delta^\circ\psi$  we obtain

$$\Delta^\circ\psi(D) \equiv \psi(1) \times \Delta \xleftarrow{\Delta^\circ\psi(\rho)} \Delta^\circ\psi(X) \xrightarrow{\Delta^\circ\psi(f)} \Delta^\circ\psi(Y),$$

so  $\tau_D : Cyl(\Delta^\circ\psi f, \Delta^\circ\psi \rho) \rightarrow \Delta^\circ\psi(Cyl(f, \rho)) = \Delta^\circ\psi(Cf)$  is a degreewise equivalence, and we deduce that  $\mathbf{s}'(\tau_D) : \mathbf{s}'Cyl(\Delta^\circ\psi f, \Delta^\circ\psi \rho) \rightarrow \mathbf{s}'\Delta^\circ\psi(Cf)$  is in  $E'$ . In addition, it follows from the commutativity of (2.22) that

$$\mathbf{s}'i_{\psi 1} : R\psi(1) \rightarrow \mathbf{s}'Cyl(\Delta^\circ\psi f, \Delta^\circ\psi \rho) \in E' \text{ if and only if } \mathbf{s}'(\Delta^\circ\psi i_1) : R\psi(1) \rightarrow \mathbf{s}'(\Delta^\circ\psi Cf) \in E'$$

By (FD 2),  $\mathbf{s}'(\Delta^\circ\psi i_1) \in E'$  if and only if  $\psi(\mathbf{s}i_1) : \psi(R1) \rightarrow \psi(\mathbf{s}Cf)$  is so.

But we proved in (SDC 5) that  $\psi(R1) \rightarrow \psi(1) \in E'$ , then

$$\mathbf{s}'i_{\psi 1} : R\psi(1) \rightarrow \mathbf{s}'Cyl(\Delta^\circ\psi f, \Delta^\circ\psi \rho) \in E' \text{ if and only if } \psi(\mathbf{s}Cf) \rightarrow \psi(1) \in E'.$$

On the other hand, if  $\rho' : Y \rightarrow 1 \times \Delta$  is the trivial morphism, we have that  $\rho' \circ f = \rho$ , and  $(\Delta^\circ\psi f) \circ (\Delta^\circ\psi \rho') = \Delta^\circ\psi \rho$ . Hence, by 2.4.5, it holds that

$$\mathbf{s}'(\Delta^\circ\psi f) \in E' \text{ if and only if } \mathbf{s}'i_{\psi(1)} : R\psi(1) \rightarrow \mathbf{s}'Cyl(\Delta^\circ\psi f, \Delta^\circ\psi \rho) \text{ is in } E',$$

so (SDC 7) is already proven.

(SDC 8) Given a morphism  $f$  in  $\Delta^\circ\mathcal{D}$  then

$$\begin{aligned} \mathbf{s}f \in E &\Leftrightarrow \psi(\mathbf{s}f) \in E' \Leftrightarrow \mathbf{s}'(\Delta^\circ\psi f) \in E' \Leftrightarrow \mathbf{s}'(\Upsilon \circ \Delta^\circ\psi f) = \mathbf{s}'(\Delta^\circ\psi(\Upsilon f)) \in E' \Leftrightarrow \\ &\Leftrightarrow \psi(\mathbf{s}(\Upsilon f)) \in E' \Leftrightarrow \mathbf{s}(\Upsilon f) \in E. \end{aligned}$$

□

**COROLLARY 2.5.9.** *If  $\mathcal{D}$  is a subcategory of a simplicial descent category  $(\mathcal{D}', E', \mathbf{s}', \mu', \lambda')$  such that*

1.  $\mathcal{D}$  is closed under coproducts
2.  $\mathcal{D}$  is closed under the simple functor, that is, if  $X \in \Delta^\circ\mathcal{D}$  then  $\mathbf{s}'(X)$  is in  $\mathcal{D}$ .

3. The value of the natural transformation  $\lambda'$  (resp.  $\mu'$ ) at any object of  $\mathcal{D}$  (resp.  $\Delta^\circ \Delta^\circ \mathcal{D}$ ) is a morphism of  $\mathcal{D}$ . For instance, if  $\mathcal{D}$  is full.

Then  $(\mathcal{D}, E' \cap \mathcal{D}, s'|_{\mathcal{D}}, \mu'|_{\mathcal{D}}, \lambda'|_{\mathcal{D}})$  is a simplicial descent category.

*Proof.* It suffices to take  $\psi = i : \mathcal{D} \rightarrow \mathcal{D}'$  in the lemma transfer.  $\square$

### 2.5.1 Associativity of $\mu$

If  $\mathcal{D}$  is a (simplicial) descent category and the natural transformation  $\mu$  is “associative” then  $\Delta^\circ \mathcal{D}$  has a second structure of descent category in addition to the one introduced in 2.1.16, this time taking the diagonal functor  $D : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  as simple functor.

The associativity property of  $\mu$  will not be used in the sequel. However, this is a relevant property and means that the descent structure can be iterated in a “suitable” way in the category of multisimplicial objects in  $\mathcal{D}$ .

Moreover, in this case the transformations  $\lambda$  and  $\mu$  give rise to a cotriple (cf. [Dus]) in  $\mathcal{D}$ .

**DEFINITION 2.5.10** (Associativity of  $\mu$ ).

Let  $D_{1,2} : \Delta^\circ \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}$  (resp.  $D_{2,3} : \Delta^\circ \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}$ ) be the functor that makes equal the two first indexes (resp. the two last indexes) of a trisimplicial object.

We will say that the natural transformation  $\mu$  is *associative* if for every  $T \in \Delta^\circ \Delta^\circ \Delta^\circ \mathcal{D}$  the following diagram commutes in  $\mathcal{D}$

$$\begin{array}{ccc} sDD_{1,2}T = sDD_{2,3}T & \xrightarrow{\mu_{D_{2,3}T}} & s\Delta^\circ sD_{2,3}T \\ \mu_{D_{1,2}T} \downarrow & & \downarrow s(\Delta^\circ \mu_T) \\ s\Delta^\circ sD_{1,2}T = sD\Delta^\circ \Delta^\circ sT & \xrightarrow{\mu_{\Delta^\circ \Delta^\circ sT}} & s_\circ \Delta^\circ s_\circ \Delta^\circ \Delta^\circ s(T) \end{array} \quad (2.23)$$

where  $\Delta^\circ \mu_T : \Delta^\circ sD_{2,3}T \rightarrow \Delta^\circ s_\circ \Delta^\circ \Delta^\circ s(T)$  is in degree  $n$  the morphism between simplicial objects

$$(\Delta^\circ \mu_T)_n = \mu_{T_{n,\cdot,\cdot}} : sDT_{n,\cdot,\cdot} \rightarrow s(\Delta^\circ sT_{n,\cdot,\cdot}) .$$

**PROPOSITION 2.5.11.** Assume that  $\mathcal{D}$  is a simplicial descent category with  $\mu$  associative. Then the data

$$\begin{aligned} R : \mathcal{D} &\rightarrow \mathcal{D} \text{ where } RX = s(X \times \Delta) \\ \lambda : R &\rightarrow Id_{\mathcal{D}} \\ \mu' : R &\rightarrow R^2 \text{ where } \mu'_X = \mu_{X \times \Delta \times \Delta} \end{aligned}$$

is a cotriple  $(R, \lambda, \mu')$  in the category  $\mathcal{D}$ .

*Proof.* The statement is an immediate consequence of the associativity of  $\mu$  and the equations describing the compatibility between  $\lambda$  and  $\mu$  (2.1).

Indeed, assume given an object  $Y$  in  $\mathcal{D}$  and set  $X = Y \times \Delta$ . Following the notations in (2.1), we have that  $\mu_{\Delta \times X \times \Delta} = \mu_{X \times \Delta \times \Delta} = \mu'_Y$ , whereas  $\lambda_{\mathbf{s}X} = \lambda_{RY}$  and  $\mathbf{s}(\lambda_{X_n}) = \mathbf{s}(\lambda_Y \times \Delta) = R\lambda_Y$ . Hence, the compatibility equations between  $\lambda$  and  $\mu$  are in this case  $\mu'_Y \circ \lambda_{RY} = \mu'_Y \circ R\lambda_Y = Id_Y$ .

On the other hand, if  $T = Y \times \Delta \times \Delta \in \Delta^\circ \Delta^\circ \mathcal{D}$  then

$$\mu_{D_{1,2}T} = \mu_{D_{2,3}T} = \mu'_Y, \quad \mu_{\Delta^\circ \Delta^\circ \mathbf{s}T} = \mu'_{RY} \text{ and } \mathbf{s}(\Delta^\circ \mu_T) = \mathbf{s}(\mu_{Y \times \Delta \times \Delta} \times \Delta) = R\mu'_Y$$

and the commutativity of diagram (2.23) in this case is just the equality  $\mu'_{RY} \circ \mu'_Y = (R\mu'_Y) \circ \mu'_Y$ .  $\square$

**PROPOSITION 2.5.12.** *If  $(\mathcal{D}, \mathbf{E}, \mathbf{s}, \mu, \lambda)$  is a simplicial descent category with  $\mu$  associative then  $\Delta^\circ \mathcal{D}$  is also a simplicial descent category, where the simple functor is the diagonal functor  $D : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  and the class of equivalences is*

$$\mathbf{E}'_{\Delta^\circ \mathcal{D}} = \{f \mid \mathbf{s}f \in \mathbf{E}\}.$$

*In addition,  $\mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$  is a functor of descent categories.*

*Proof.* The result follows from the transfer lemma, setting  $\psi = \mathbf{s} : \Delta^\circ \mathcal{D} \rightarrow \mathcal{D}$ . Axioms (SDC 1) and (SDC 3)' hold. The natural transformations  $\mu_{\Delta^\circ \mathcal{D}}$  and  $\lambda_{\Delta^\circ \mathcal{D}}$  are both the identity natural transformation, therefore they satisfy trivially the equalities (2.1).

On the other hand, (FD 1) is a consequence of the additivity axiom, whereas by the normalization one, the natural transformation  $\Theta = \mu : \mathbf{s}D \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}$  is a (pointwise) equivalence.

If  $X$  is a simplicial object in  $\mathcal{D}$ , diagram (2.18) is just

$$\begin{array}{ccc} \mathbf{s}D(X \times \Delta) & \xrightarrow{\mathbf{s}(Id)=Id} & \mathbf{s}X \\ \downarrow \mu_{X \times \Delta} & \nearrow \lambda_{\mathbf{s}X} & \\ \mathbf{s}((\mathbf{s}X) \times \Delta) & & \end{array}$$

that commutes in  $\mathcal{D}$  by the compatibility condition between  $\lambda$  and  $\mu$ .

Consider now  $T \in \Delta^\circ \Delta^\circ \Delta^\circ \mathcal{D}$ . Under this setting, the usual diagonal functor  $D : \Delta^\circ \Delta^\circ (\Delta^\circ \mathcal{D}) \rightarrow \Delta^\circ \mathcal{D}$  is  $D_{1,2}$ , whereas  $\Delta^\circ D : \Delta^\circ \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}$  is by definition  $D_{2,3}$ .

Hence, diagram (2.19) can be written as

$$\begin{array}{ccc}
\mathbf{sD}_{1,2}(\Delta^\circ \Delta^\circ \mathbf{s}T) & \xrightarrow{\mu_{\Delta^\circ \Delta^\circ \mathbf{s}T}} & \mathbf{s}\Delta^\circ \mathbf{s}(\Delta^\circ \Delta^\circ \mathbf{s}T) \\
\uparrow \mu_{\mathbf{D}_{1,2}T} & & \uparrow (\mathbf{s}\Delta^\circ \mu)_T \\
& & \mathbf{s}\Delta^\circ \mathbf{sD}_{2,3}T \\
& & \uparrow \mu_{\mathbf{D}_{2,3}T} \\
\mathbf{sDD}_{1,2}T & \xrightarrow{\mathbf{s}(Id)=Id} & \mathbf{sDD}_{2,3}T
\end{array}$$

so its commutativity is just the associativity condition satisfied by  $\mu$ .  $\square$

## Chapter 3

# The homotopy category of a simplicial descent category

### 3.1 Description of $Ho\mathcal{D}$

This section is devoted to the study of the homotopy category associated with a simplicial category  $\mathcal{D}$ , that is by definition  $\mathcal{D}[E^{-1}]$ .

In general the class  $E$  does not has calculus of fractions, for instance when  $\mathcal{D}$  =chain complexes and  $E$  =morphisms inducing isomorphism in homology. However, some of the properties satisfied by the functor  $cyl$  developed in the last chapter are similar, but in a more general sense, to the left calculus of fractions (or to the right calculus of fractions in the cosimplicial case). This fact will allow us to exhibit a “reasonable” description of the morphisms in  $Ho\mathcal{D}$ . From now on  $\mathcal{D}$  will be a simplicial descent category.

**(3.1.1)** Let  $X$  be an object in  $\mathcal{D}$ . We remind that  $R : \mathcal{D} \rightarrow \mathcal{D}$  is  $RX = s(X \times \Delta)$ . In addition, if  $T = X \times \Delta \in \Delta^\circ \Delta^\circ \mathcal{D}$  then  $\mu_T : RX \rightarrow R^2X$ . Denote also by  $\mu : R \rightarrow R^2$  the natural transformation obtained in this way, that is,  $\mu_X$  means  $\mu_{X \times \Delta \times \Delta}$ .

From the compatibility between  $\lambda$  and  $\mu$  (2.1) we deduce that the following compositions must be the identity in  $\mathcal{D}$

$$\begin{aligned} RX &\xrightarrow{\mu_X} R^2X \xrightarrow{\lambda_{RX}} RX \\ RX &\xrightarrow{\mu_X} R^2X \xrightarrow{R\lambda_X} RX . \end{aligned} \tag{3.1}$$

Note also that from the naturality of  $\lambda : R \rightarrow Id_{\mathcal{D}}$  it follows that  $\lambda_X \circ \lambda_{RX} = \lambda_X \circ R\lambda_X$ .

**DEFINITION 3.1.2.** Let  $Ho\mathcal{D}$  be the category with the same objects as  $\mathcal{D}$  and whose morphisms are described as follows.  
Given objects  $X, Y$  in  $\mathcal{D}$  then

$$Hom_{Ho\mathcal{D}}(X, Y) = T(X, Y) / \sim$$

where an element  $F$  of  $T(X, Y)$  is a zig-zag

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y, \quad w \in E.$$

If  $X \xleftarrow{\lambda_X} RX \xrightarrow{g} S \xleftarrow{u} RY \xrightarrow{\lambda_Y} Y$  is another element  $G \in T(X, Y)$ , then  $F$  is related to  $G$ ,  $F \sim G$ , if and only there exists a ‘hammock’ (that is, a commutative diagram in  $\mathcal{D}$ )

$$\begin{array}{ccccccc} & & R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y \\ & \swarrow Id & \downarrow & & \downarrow & & \downarrow Id \\ R^2X & \xleftarrow{\quad} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{\quad} & \tilde{Y} \xrightarrow{\quad} R^2Y \\ & \nwarrow Id & \uparrow & & \uparrow & & \uparrow Id \\ & & R^2X & \xrightarrow{Rg} & RS & \xleftarrow{Ru} & R^2Y \end{array}, \quad (3.2)$$

relating  $F$  and  $G$ , such that all maps except  $f, g$  and  $h$  are equivalences.

**(3.1.3)** Given two composable morphisms in  $Ho\mathcal{D}$  represented by zig-zags  $F$  and  $G$  given respectively by

$$\begin{aligned} X &\xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{u} RY \xrightarrow{\lambda_Y} Y \\ Y &\xleftarrow{\lambda_Y} RY \xrightarrow{g} S \xleftarrow{v} RZ \xrightarrow{\lambda_Z} Z, \end{aligned}$$

then their composition is represented by the zig-zag  $G \circ F$  defined as

$$X \xleftarrow{\lambda_X} RX \xrightarrow{h} cyl(u, g) \xleftarrow{w} RZ \xrightarrow{\lambda_Z} Z,$$

where the morphisms  $h : RX \rightarrow cyl(u, g)$  and  $w : RZ \rightarrow cyl(u, g) \in E$  are the respective compositions

$$\begin{aligned} RX &\xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RT \xrightarrow{I_T} cyl(u, g) \\ RZ &\xrightarrow{\mu_Z} R^2Z \xrightarrow{Rv} RS \xrightarrow{I_S} cyl(u, g). \end{aligned}$$

By 2.4.1,  $I_S \in E$ . Therefore  $w \in E$  since it is the composition of two equivalences.

**REMARK 3.1.4.** Note that if we compose the hammock (3.2) with  $\lambda$  we get the following hammock where the upper zig-zag is  $F$  and the lower one is  $G$

$$\begin{array}{ccccccc}
 & & RX & \xrightarrow{f} & T & \xleftarrow{w} & RY \\
 & \nearrow \lambda_X & \uparrow \lambda_{RX} & & \uparrow \lambda_T & & \uparrow \lambda_{RY} \\
 & & R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y \\
 & \nearrow \lambda_X & \downarrow & & \downarrow & & \downarrow \lambda_Y \\
 X & \xleftarrow{\lambda_X} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{\lambda_Y} & \tilde{Y} & \xrightarrow{\lambda_Y} & Y \\
 & \nwarrow \lambda_X & \uparrow & & \uparrow & & \uparrow & & \\
 & & R^2X & \xrightarrow{Rg} & RS & \xleftarrow{Ru} & R^2Y \\
 & \nwarrow \lambda_X & \downarrow \lambda_{RX} & & \downarrow \lambda_S & & \downarrow \lambda_{RY} \\
 & & RX & \xrightarrow{g} & S & \xleftarrow{u} & RY
 \end{array} \quad (3.3)$$

**THEOREM 3.1.5.**  $Ho\mathcal{D}$  is in fact a category. Moreover, the functor

$$\gamma : \mathcal{D} \rightarrow Ho\mathcal{D}$$

defined as the identity over objects, and over morphisms as

$$\gamma(X \xrightarrow{f} Y) = X \xleftarrow{\lambda_X} RX \xrightarrow{Rf} Y \xleftarrow{Id} RY \xrightarrow{\lambda_Y} Y$$

is a localization of  $\mathcal{D}$  with respect to  $E$ .

The proof of the above theorem is very similar to the analogue proof in the calculus of fractions case, except that now a great number of technical problems must be solved. This is why we decide to divide this proof into the following lemmas and preliminary results.

**LEMMA 3.1.6.** Two elements  $F$  and  $G$  in  $T(X, Y)$  given by

$$\begin{array}{l}
 X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y \\
 X \xleftarrow{\lambda_X} RX \xrightarrow{g} S \xleftarrow{u} RY \xrightarrow{\lambda_Y} Y
 \end{array}$$

are such that  $F \sim G$  if and only if there exists  $k \geq 0$  and a hammock  $\mathcal{H}^k$

$$\begin{array}{ccccccc}
 & & R^{k+1}X & \xrightarrow{R^k f} & R^k T & \xleftarrow{R^k w} & R^{k+1}Y \\
 & \nearrow Id & \downarrow & & \downarrow & & \downarrow Id \\
 R^{k+1}X & \xleftarrow{Id} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{Id} & \tilde{Y} & \xrightarrow{Id} & R^{k+1}Y \\
 & \nwarrow Id & \uparrow & & \uparrow & & \uparrow & & \\
 & & R^{k+1}X & \xrightarrow{R^k g} & R^k S & \xleftarrow{R^k u} & R^{k+1}Y
 \end{array}$$

where all maps except  $f$ ,  $g$  and  $h$  are equivalences.

*Proof.* If  $k = 0$ , it is enough to apply  $R$  to the hammock  $\mathcal{H}^0$ , since  $R(E) \subseteq E$ . If  $k > 1$ , every hammock  $H^k$  gives rise to a new one with  $k = 1$ , through the natural transformations  $\lambda^k : R^k \rightarrow R$  and  $\mu^k : R \rightarrow R^k$ .

More specifically, let  $\lambda^n$  be the natural transformation defined as  $\lambda_X = \lambda_{R^{k-1}X} \circ \lambda_{R^{k-1}X} \circ \cdots \circ \lambda_{RX} : R^k X \rightarrow RX$ , as well as  $\mu_X^k = \mu_{R^{k-2}X} \circ \cdots \circ \mu_X : R \rightarrow R^k$ .

Then, from (3.1) we deduce that the composition

$$R^k X \xrightarrow{\mu_X^k} RX \xrightarrow{\lambda_X^k} R^k X$$

is equal to the identity.

Let us do the computation for the upper half of  $\mathcal{H}^k$ , since it can be argued similarly for the lower one. To this end, just note that the naturality of  $\mu^k$  together with the equality  $\lambda^k \circ \mu^k = Id$  imply that the following diagram is commutative

$$\begin{array}{ccccccc} & & R^2 X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2 Y \\ & & \downarrow \mu_{RX}^k & & \downarrow \mu_T^k & & \downarrow \mu_{RY}^k \\ & & R^{k+1} X & \xrightarrow{R^k f} & R^k T & \xleftarrow{R^k w} & R^{k+1} Y \\ & \swarrow Id & \downarrow Id & & \downarrow Id & \searrow Id & \\ R^2 X & \xleftarrow{\lambda_{RX}^k} & R^{k+1} X & \xleftarrow{\quad} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{\quad} & \tilde{Y} & \xrightarrow{\quad} & R^{k+1} Y & \xrightarrow{\lambda_{RY}^k} & R^2 Y \end{array}$$

□

**LEMMA 3.1.7.** *The relation  $\sim$  is an equivalence relation over  $T(X, Y)$ .*

*Proof.* The relation  $\sim$  is symmetric by definition. The reflexivity is also clear, just take the vertical morphisms in the corresponding hammock as identities.

It remains to check that  $\sim$  is transitive. Assume that  $F \sim G$  and  $G \sim L$  through the hammocks  $\mathcal{H}$  and  $\mathcal{H}'$  given respectively by

$$\begin{array}{ccccccc} & & R^2 X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2 Y \\ & \swarrow Id & \downarrow & & \downarrow & & \downarrow Id \\ & R^2 X & \xleftarrow{\alpha} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{u} & \tilde{Y} & \xrightarrow{\beta} & R^2 Y \\ & \swarrow Id & \downarrow p & & \downarrow p'' & & \downarrow p' & & \downarrow Id \\ & R^2 X & \xrightarrow{Rg} & RS & \xleftarrow{Rw'} & R^2 Y \end{array} \quad ; \quad \begin{array}{ccccccc} & & R^2 X & \xrightarrow{Rg} & RS & \xleftarrow{Rw'} & R^2 Y \\ & \swarrow Id & \downarrow q & & \downarrow q'' & & \downarrow q' Id \\ & R^2 X & \xleftarrow{\alpha'} & \hat{X} & \xrightarrow{h'} & W & \xleftarrow{u'} & \hat{Y} & \xrightarrow{\beta'} & R^2 Y \\ & \swarrow Id & \downarrow & & \downarrow & & \downarrow & & \downarrow Id \\ & R^2 X & \xrightarrow{Rl} & RV & \xleftarrow{Rw''} & R^2 Y \end{array}$$



Applying by columns the functor  $cyl$  we get

$$\begin{array}{ccccc}
\tilde{X} & \xrightarrow{h} & U & \xleftarrow{u} & \tilde{Y} \\
p \uparrow & & p'' \uparrow & & p' \uparrow \\
R^2 X & \xrightarrow{Rg} & RS & \xleftarrow{Rw'} & R^2 Y \\
q \downarrow & & q'' \downarrow & & q' \downarrow \\
\hat{X} & \xrightarrow{h'} & W & \xleftarrow{u'} & \hat{Y}
\end{array}$$

and setting  $\overline{X} = cyl(q, p)$ ,  $M = cyl(q'', p'')$  and  $\overline{Y} = cyl(q', p')$  we obtain the commutative diagram in  $\mathcal{D}$

$$\begin{array}{ccccc}
R\tilde{X} & \xrightarrow{Rh} & RU & \xleftarrow{Ru} & R\tilde{Y} \\
s \downarrow & & \downarrow & & s' \downarrow \\
\overline{X} & \longrightarrow & M & \longleftarrow & \overline{Y} \\
t \uparrow & & \uparrow & & t' \uparrow \\
R\hat{X} & \xrightarrow{Rh'} & RW & \xleftarrow{Ru'} & R\hat{Y}
\end{array} \tag{3.4}$$

where all vertical arrows are equivalences by 2.4.1, as well as  $\overline{Y} \rightarrow M$  by 2.2.6. In addition, since  $\alpha \circ p = \alpha' \circ q = Id_{R^2 X}$  (resp.  $\beta \circ p' = \beta' \circ q' = Id_{R^2 Y}$ ), it follows from 2.2.9 the existence of a morphism  $\rho : \overline{X} \rightarrow R^3 X$  (resp.  $\rho' : \overline{Y} \rightarrow R^3 Y$ ) such that  $\rho \circ s = R\alpha$  and  $\rho \circ t = R\alpha'$  (resp.  $\rho' \circ s' = R\beta$  and  $\rho' \circ t' = R\beta'$ ).

By the 2 out of 3 property we have that  $\rho$  (resp.  $\rho'$ ) is an equivalence.

On the other hand, applying  $R$  to the upper half of  $\mathcal{H}$  and to the lower half of  $\mathcal{H}'$  we obtain

$$\begin{array}{ccccccc}
& & R^3 X & \xrightarrow{R^2 f} & R^2 T & \xleftarrow{R^2 w} & R^3 Y \\
& \swarrow Id & \downarrow & & \downarrow & & \downarrow \\
R^3 X & \xleftarrow{R\alpha} & R\tilde{X} & \xrightarrow{Rh} & RU & \xleftarrow{Ru} & R\tilde{Y} \\
& & & & & & \downarrow \\
& & & & & & R^3 Y \\
& \searrow Id & & & & & \\
& & R^3 X & \xrightarrow{R\alpha'} & R\hat{X} & \xrightarrow{Rh'} & RW \\
& & & & & & \downarrow \\
& & & & & & R\hat{Y} \\
& & & & & & \downarrow \\
& & & & & & R^3 Y \\
& \swarrow Id & & & & & \\
& & R^3 X & \xrightarrow{R^2 l} & R^2 V & \xleftarrow{R^2 w''} & R^3 Y \\
& & & & & & \downarrow \\
& & & & & & R^3 Y
\end{array}$$

and after adjoining them to (3.4) and composing, the result is

$$\begin{array}{ccccccc}
& & R^3 X & \xrightarrow{R^2 f} & R^2 T & \xleftarrow{R^2 w} & R^3 Y \\
& \swarrow Id & \downarrow & & \downarrow & & \downarrow \\
R^3 X & \xleftarrow{\rho} & \overline{X} & \longrightarrow & M & \longleftarrow & \overline{Y} \\
& & \uparrow & & \uparrow & & \uparrow \\
& & R^3 X & \xrightarrow{R^2 l} & R^2 V & \xleftarrow{R^2 w''} & R^3 Y \\
& \swarrow Id & & & & & \downarrow \\
& & & & & & R^3 Y
\end{array}$$

Then, by the previous lemma,  $F \sim L$ . □

**LEMMA 3.1.8.** *The composition of morphisms in  $Ho\mathcal{D}$  is well defined, that is, if  $F \sim F'$  and  $G \sim G'$  represent two composable morphisms in  $Ho\mathcal{D}$  then  $G \circ F \sim G' \circ F'$ .*

*Proof.* Assume that  $F \sim F'$  and  $G \sim G'$  through the hammocks  $\mathcal{H}$  and  $\mathcal{H}'$  given respectively by

$$\begin{array}{ccccc}
 R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y \\
 \swarrow Id & \downarrow s & \downarrow s'' & \downarrow s' & \searrow Id \\
 R^2X & \xleftarrow{\alpha} & \tilde{X} & \xrightarrow{h} & L & \xleftarrow{u} & \tilde{Y} & \xrightarrow{\beta} & R^2Y \\
 \swarrow Id & \downarrow p & \downarrow p'' & \downarrow p' & \searrow Id \\
 R^2X & \xrightarrow{Rf'} & RS & \xleftarrow{Rw'} & R^2Y
 \end{array}
 \quad ; \quad
 \begin{array}{ccccc}
 R^2Y & \xrightarrow{Rg} & RU & \xleftarrow{Rv} & R^2Z \\
 \swarrow Id & \downarrow q & \downarrow q'' & \downarrow q' & \searrow Id \\
 R^2Y & \xleftarrow{\alpha'} & \hat{Y} & \xrightarrow{h'} & W & \xleftarrow{u'} & \hat{Z} & \xrightarrow{\beta'} & R^2Z \\
 \swarrow Id & \downarrow t & \downarrow t'' & \downarrow t' & \searrow Id \\
 R^2Y & \xrightarrow{Rg'} & RV & \xleftarrow{Rv'} & R^2Z
 \end{array}$$

We have to find a hammock relating the zig-zags

$$\begin{aligned}
 X &\xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RT \xrightarrow{I_T} cyl(w, g) \xleftarrow{I_U} RU \xleftarrow{Rv} R^2Z \xleftarrow{\mu_Z} RZ \xrightarrow{\lambda_Z} Z \\
 X &\xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf'} RS \xrightarrow{I_S} cyl(w', g') \xleftarrow{I_V} RV \xleftarrow{Rv'} R^2Z \xleftarrow{\mu_Z} RZ \xrightarrow{\lambda_Z} Z .
 \end{aligned}$$

To this end, apply by rows the functor  $cyl$  to the diagram

$$\begin{array}{ccccc}
 RT & \xleftarrow{Rw} & R^2Y & \xrightarrow{Id} & R^2Y \\
 \downarrow s'' & & \downarrow & & \downarrow Id \\
 L & \xleftarrow{u} & \tilde{Y} & \xrightarrow{\beta} & R^2Y \\
 \uparrow p'' & & \uparrow p' & & \uparrow Id \\
 RS & \xleftarrow{Rw'} & R^2Y & \xrightarrow{Id} & R^2Y
 \end{array}$$

Then we obtain

$$\begin{array}{ccccc}
 R^2T & \xrightarrow{J_T} & cyl(Rw, Id) & \xleftarrow{J_Y} & R^3Y \\
 \downarrow Rs'' & & \downarrow & & \downarrow Id \\
 RL & \longrightarrow & cyl(u, \beta) & \longleftarrow & R^3Y \\
 \uparrow Rp'' & & \uparrow & & \uparrow Id \\
 R^2S & \xrightarrow{J_S} & cyl(Rw', Id) & \xleftarrow{I_Y} & R^3Y
 \end{array}$$

where all arrows are equivalences, by properties 2.4.1 and 2.2.6 of  $cyl$ .

On the other hand, since  $Id_{RT} \circ Rw = Rw \circ Id_{R^2Y}$  (resp.  $Id_{RS} \circ Rw' = Rw' \circ Id_{R^2Y}$ ), it follows from 2.2.9 the existence of a morphism  $\rho : cyl(Rw, Id) \rightarrow R^2T$  (resp.  $\rho' : cyl(Rw', Id) \rightarrow R^2S$ ) such that  $\rho \circ J_T = Id_{R^2T}$  and  $\rho \circ J_Y = R^2w$  (resp.

$\rho' \circ J_S = Id_{R^2S}$  and  $\rho' \circ I_Y = R^2w'$ ).

From the 2 out of 3 property we deduce that  $\rho$  and  $\rho'$  are equivalences. So we can construct the following diagram from these two maps, together with the result of applying  $R$  to some parts of  $\mathcal{H}$  and  $\mathcal{H}'$ , getting

$$\begin{array}{ccccccccc}
R^3X & \xrightarrow{R^2f} & R^2T & \xrightarrow{Id} & R^2T & \xleftarrow{R^2w} & R^3Y & \xleftarrow{Id} & R^3Y & \xrightarrow{R^2g} & R^2U \\
Id \uparrow & & \uparrow Id & & \rho \uparrow & & Id \uparrow & & Id \uparrow & & Id \uparrow \\
R^3X & \xrightarrow{R^2f} & R^2T & \xrightarrow{J_T} & cyl(Rw, Id) & \xleftarrow{J_Y} & R^3Y & \xleftarrow{Id} & R^3Y & \xrightarrow{R^2g} & R^2U \\
Rs \downarrow & & \downarrow Rs & & \downarrow & & Id \downarrow & & Rq \downarrow & & Rq'' \downarrow \\
R\tilde{X} & \xrightarrow{Rh} & RL & \longrightarrow & cyl(u, \beta) & \longleftarrow & R^3Y & \xleftarrow{R\alpha'} & R\hat{Y} & \xrightarrow{Rh'} & RW \\
Rp \uparrow & & \uparrow Rp'' & & \uparrow & & Id \uparrow & & Rt \uparrow & & Rt'' \uparrow \\
R^3X & \xrightarrow{R^2f'} & R^2S & \xrightarrow{J_S} & cyl(Rw', Id) & \xleftarrow{I_Y} & R^3Y & \xleftarrow{Id} & R^3Y & \xrightarrow{R^2g'} & R^2V \\
Id \downarrow & & \downarrow Id & & \rho' \downarrow & & Id \downarrow & & Id \downarrow & & Id \downarrow \\
R^3X & \xrightarrow{R^2f'} & R^2S & \xrightarrow{Id} & R^2S & \xleftarrow{R^2w'} & R^3Y & \xleftarrow{Id} & R^3Y & \xrightarrow{R^2g'} & R^2V .
\end{array}$$

Composing arrows in the above diagram and attaching the remaining part of  $\mathcal{H}'$  we get

$$\begin{array}{ccccccccc}
R^3X & \xrightarrow{R^2f} & R^2T & \xleftarrow{R^2w} & R^3Y & \xrightarrow{R^2g} & R^2U & \xleftarrow{R^2v} & R^3Z \\
Id \uparrow & & \rho \uparrow & & Id \uparrow & & Id \uparrow & & Id \uparrow \\
R^3X & \longrightarrow & cyl(Rw, Id) & \longleftarrow & R^3Y & \xrightarrow{R^2g} & R^2U & \xleftarrow{R^2v} & R^3Z \\
Rs \downarrow & & \downarrow & & Rq \downarrow & & Rq'' \downarrow & & Rq' \downarrow \\
R\tilde{X} & \longrightarrow & cyl(u, \beta) & \longleftarrow & R\hat{Y} & \xrightarrow{Rh'} & RW & \xleftarrow{Ru'} & R\hat{Z} \\
Rp \uparrow & & \uparrow & & Rt \uparrow & & Rt'' \uparrow & & Rt' \uparrow \\
R^3X & \longrightarrow & cyl(Rw', Id) & \longleftarrow & R^3Y & \xrightarrow{R^2g'} & R^2V & \xleftarrow{R^2v'} & R^3Z \\
Id \downarrow & & \rho' \downarrow & & Id \downarrow & & Id \downarrow & & Id \downarrow \\
R^3X & \xrightarrow{R^2f'} & R^2S & \xleftarrow{R^2w'} & R^3Y & \xrightarrow{R^2g'} & R^2V & \xleftarrow{R^2v'} & R^3Z
\end{array}$$

where all vertical maps are equivalences, as well as the columns that contains  $R^2w$  and  $R^2v$ . Now compose with the natural transformation  $\lambda^2 : R^2 \rightarrow Id$  to obtain

$$\begin{array}{ccccccccc}
RX & \xrightarrow{f} & T & \xleftarrow{w} & RY & \xrightarrow{g} & U & \xleftarrow{v} & RZ \\
\lambda_{RX}^2 \uparrow & & \varrho \uparrow & & \lambda_{RY}^2 \uparrow & & \lambda_U^2 \uparrow & & \lambda_{RZ}^2 \uparrow \\
R^3X & \longrightarrow & cyl(Rw, Id) & \longleftarrow & R^3Y & \xrightarrow{R^2g} & R^2U & \xleftarrow{R^2v} & R^3Z \\
Rs \downarrow & & \downarrow & & Rq \downarrow & & Rq'' \downarrow & & Rq' \downarrow \\
R\tilde{X} & \longrightarrow & cyl(u, \beta) & \longleftarrow & R\hat{Y} & \xrightarrow{Rh'} & RW & \xleftarrow{Ru'} & R\hat{Z} \\
Rp \uparrow & & \uparrow & & Rt \uparrow & & Rt'' \uparrow & & Rt' \uparrow \\
R^3X & \longrightarrow & cyl(Rw', Id) & \longleftarrow & R^3Y & \xrightarrow{R^2g'} & R^2V & \xleftarrow{R^2v'} & R^3Z \\
\lambda_{RX}^2 \downarrow & & \varrho' \downarrow & & \lambda_{RY}^2 \downarrow & & \lambda_V^2 \downarrow & & \lambda_{RZ}^2 \downarrow \\
RX & \xrightarrow{f'} & S & \xleftarrow{w'} & RY & \xrightarrow{g'} & V & \xleftarrow{v'} & RZ
\end{array}$$

where  $\varrho = \lambda_{T \circ \rho}$  and  $\varrho' = \lambda_{S \circ \rho'}$ . The result of applying  $cyl$  to the two middle columns in the above diagram is

$$\begin{array}{ccccccc}
R^2 X & \xrightarrow{Rf} & RT & \xrightarrow{I_T} & cyl(w, g) & \xleftarrow{I_U} & RU \xleftarrow{Rv} R^2 Z \\
\uparrow R\lambda_{RX}^2 & & \uparrow R\varrho & & \uparrow & & \uparrow R\lambda_U^2 \uparrow R\lambda_{R^2 Z}^2 \\
R^4 X & \rightarrow & R cyl(Rw, Id) & \rightarrow & M' & \leftarrow & R^3 U \xleftarrow{R^3 v} R^4 Z \\
\downarrow R^2 s & & \downarrow & & \downarrow & & \downarrow R^2 q' \\
R^2 \tilde{X} & \rightarrow & R cyl(u, \beta) & \rightarrow & M & \leftarrow & R^2 W \xleftarrow{R^2 u'} R^2 \hat{Z} \\
\uparrow R^2 p & & \uparrow & & \uparrow & & \uparrow R^2 t' \\
R^4 X & \rightarrow & R cyl(Rw', Id) & \rightarrow & M'' & \leftarrow & R^3 V \xleftarrow{R^3 v'} R^4 Z \\
\downarrow R\lambda_{RX}^2 & & \downarrow R\varrho' & & \downarrow & & \downarrow R\lambda_V^2 \downarrow R\lambda_{R^2 Z}^2 \\
R^2 X & \xrightarrow{Rf'} & RS & \xrightarrow{I_S} & cyl(w', g') & \xleftarrow{I_V} & RV \xleftarrow{Rv'} R^2 Z
\end{array}$$

Now apply  $cyl$  to the three first rows, getting

$$\begin{array}{ccccccc}
R^3 X & \xrightarrow{R^2 f} & R^2 T & \xrightarrow{RI_T} & R cyl(w, g) & \xleftarrow{RI_U} & R^2 U \xleftarrow{R^2 v} R^3 Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
cyl(R\lambda_{RX}^2, R^2 s) & \rightarrow & N' & \rightarrow & N & \leftarrow & N'' \leftarrow cyl(R\lambda_{R^2 Z}^2, R^2 q') \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R^3 \tilde{X} & \rightarrow & R^2 cyl(u, \beta) & \rightarrow & RM & \leftarrow & R^3 W \xleftarrow{R^3 u'} R^3 \hat{Z} \\
\uparrow R^3 p & & \uparrow & & \uparrow & & \uparrow R^3 t' \\
R^5 X & \rightarrow & R^2 cyl(Rw', Id) & \rightarrow & RM'' & \leftarrow & R^4 V \xleftarrow{R^4 v'} R^5 Z \\
\downarrow R^2 \lambda_{RX}^2 & & \downarrow R^2 \varrho' & & \downarrow & & \downarrow R^2 \lambda_{R^2 Z}^2 \\
R^3 X & \xrightarrow{R^2 f'} & R^2 S & \xrightarrow{RI_S} & R cyl(w', g') & \xleftarrow{RI_V} & R^2 V \xleftarrow{R^2 v'} R^3 Z
\end{array} \tag{3.5}$$

On the other hand, since  $R^2 \alpha \circ R^2 s = Id_{R^4 X}$  and  $R^2 \beta' \circ R^2 q' = Id_{R^4 Z}$ , the equivalences  $\sigma = R\lambda_{RX}^2 \circ R^2 \alpha$  and  $\sigma' = R\lambda_{RZ}^2 \circ R^2 \beta'$  fit into the commutative diagrams of  $\mathcal{D}$

$$\begin{array}{ccc}
R^4 X & \xrightarrow{R\lambda_{RX}^2} & R^2 Z \\
R^2 s \downarrow & & Id \downarrow \\
R^2 \tilde{X} & \xrightarrow{\sigma} & R^2 Z
\end{array}
\quad
\begin{array}{ccc}
R^4 Z & \xrightarrow{R\lambda_{RZ}^2} & R^2 Z \\
R^2 q' \downarrow & & Id \downarrow \\
R^2 \hat{Z} & \xrightarrow{\sigma'} & R^2 Z .
\end{array}$$

It follows the existence of equivalences  $\delta$  and  $\delta'$  such that the diagrams

$$\begin{array}{ccc}
& & R^3 X & & \\
& Id \nearrow & \uparrow \delta & \nwarrow R\sigma & \\
R^3 X & \rightarrow & cyl(R\lambda_{RX}^2, R^2 s) & \leftarrow & R^3 \tilde{X}
\end{array}
\quad
\begin{array}{ccc}
& & R^3 Z & & \\
& Id \nearrow & \uparrow \delta' & \nwarrow R\sigma' & \\
R^3 Z & \rightarrow & cyl(R\lambda_{R^2 Z}^2, R^2 q') & \leftarrow & R^3 \hat{Z} .
\end{array}$$

are commutative. In addition, by definition we have that  $\sigma \circ R^2 p = R\lambda_{RX}^2$  and  $\sigma' \circ R^2 t' = R\lambda_{RZ}^2$ .

If  $A = cyl(R\lambda_{RX}^2, R^2 s)$  and  $A' = cyl(R\lambda_{RZ}^2, R^2 q')$ , attaching these data to (3.5) and composing arrows we obtain

$$\begin{array}{ccccccccccc}
R^3 X & \xleftarrow{Id} & R^3 X & \xrightarrow{R^2 f} & R^2 T & \xrightarrow{RI_T} & Rcyl(w, g) & \xleftarrow{RI_U} & R^2 U & \xleftarrow{R^2 v} & R^3 Z & \xrightarrow{Id} & R^3 Z \\
Id \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow Id \\
R^3 X & \xleftarrow{\delta} & A & \longrightarrow & N' & \longrightarrow & N & \longleftarrow & N'' & \longleftarrow & A' & \xrightarrow{\sigma'} & R^3 Z \\
Id \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow R^3 t'' & & \uparrow R^3 t' & & \uparrow Id \\
R^3 X & \xleftarrow{R^2 \lambda_{RX}^2} & R^5 X & \longrightarrow & R^2 cyl(Rw', Id) & \longrightarrow & RM'' & \longleftarrow & R^4 V & \xleftarrow{R^4 v'} & R^5 Z & \xrightarrow{R^2 \lambda_{RZ}^2} & R^3 Z \\
Id \downarrow & & R^2 \lambda_{RX}^2 \downarrow & & R^2 \varrho' \downarrow & & \downarrow & & R^2 \lambda_V^2 \downarrow & & \downarrow R^2 \lambda_{RZ}^2 & & \downarrow Id \\
R^3 X & \xleftarrow{Id} & R^3 X & \xrightarrow{R^2 f'} & R^2 S & \xrightarrow{RI_S} & Rcyl(w', g') & \xleftarrow{RI_V} & R^2 V & \xleftarrow{R^2 v'} & R^3 Z & \xrightarrow{Id} & R^3 Z .
\end{array}$$

Finally, apply  $cyl$  to the three lower rows to get

$$\begin{array}{ccccccccccc}
R^4 X & \xleftarrow{Id} & R^4 X & \xrightarrow{R^3 f} & R^3 T & \xrightarrow{R^2 I_T} & R^2 cyl(w, g) & \xleftarrow{R^2 I_U} & R^3 U & \xleftarrow{R^3 v} & R^4 Z & \xrightarrow{Id} & R^4 Z \\
Id \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow Id \\
R^4 X & \xleftarrow{R\delta} & RA & \longrightarrow & RN' & \longrightarrow & RN & \longleftarrow & RN'' & \longleftarrow & RA' & \xrightarrow{R\sigma'} & R^4 Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
cyl(R^3 X) & \xleftarrow{} & \widehat{B} & \longrightarrow & B' & \longrightarrow & B & \longleftarrow & B'' & \longleftarrow & \widetilde{B} & \longrightarrow & cyl(R^3 Z) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R^4 X & \xleftarrow{Id} & R^4 X & \xrightarrow{R^3 f'} & R^3 S & \xrightarrow{R^2 I_S} & R^2 cyl(w', g') & \xleftarrow{R^2 I_V} & R^3 V & \xleftarrow{R^3 v'} & R^4 Z & \xrightarrow{Id} & R^4 Z .
\end{array} \tag{3.6}$$

Let  $\eta$  and  $\eta'$  be the equivalences deduced from 2.2.9. Then the diagrams

$$\begin{array}{ccc}
& R^4 X & \\
Id \nearrow & \uparrow \eta & \nwarrow Id \\
R^4 X & \longrightarrow cyl(R^3 X) & \longleftarrow R^4 X
\end{array}
\quad
\begin{array}{ccc}
& R^4 Z & \\
Id \nearrow & \uparrow \eta' & \nwarrow Id \\
R^4 Z & \longrightarrow cyl(R^3 Z) & \longleftarrow R^4 Z
\end{array}$$

commute. If we adjoin  $\eta$  and  $\eta'$  to (3.6) and compose arrows, the result is the hammock

$$\begin{array}{ccccccc}
& R^4 X & \xrightarrow{R^3 f} & R^3 T & \xrightarrow{R^2 I_T} & R^2 cyl(w, g) & \xleftarrow{R^2 I_U} & R^3 U & \xleftarrow{R^3 v} & R^4 Z \\
& \swarrow Id & \downarrow m & \downarrow l' & \downarrow l & \downarrow l'' & \downarrow l'' & \downarrow k & \searrow Id \\
R^4 X & \xleftarrow{\tau} & \widehat{B} & \longrightarrow & B' & \longrightarrow & B & \longleftarrow & B'' & \longleftarrow & \widetilde{B} & \xrightarrow{\tau'} & R^4 Z , \\
& \swarrow Id & \downarrow m' & \downarrow r' & \downarrow r & \downarrow r'' & \downarrow r'' & \downarrow k' & \searrow Id \\
& R^4 X & \xrightarrow{R^3 f'} & R^3 S & \xrightarrow{R^2 I_S} & R^2 cyl(w', g') & \xleftarrow{R^2 I_V} & R^3 V & \xleftarrow{R^3 v'} & R^4 Z
\end{array}$$

that gives rise to

$$\begin{array}{ccccccccccccccc}
& & \mathbf{R}^3 X & \xrightarrow{\mathbf{R}^2 \mu_X} & \mathbf{R}^4 X & \xrightarrow{\mathbf{R}^3 f} & \mathbf{R}^3 T & \xrightarrow{\mathbf{R}^2 I_T} & \mathbf{R}^2 cyl(w, g) & \xleftarrow{\mathbf{R}^2 I_U} & \mathbf{R}^3 U & \xleftarrow{\mathbf{R}^3 v} & \mathbf{R}^4 Z & \xleftarrow{\mathbf{R}^2 \mu_Z} & \mathbf{R}^3 Z \\
& \swarrow Id & \downarrow & & \downarrow m & & \downarrow l' & & \downarrow l & & \downarrow l'' & & \downarrow k & & \downarrow \\
\mathbf{R}^3 X & \xleftarrow{\quad} & \widehat{B} & \xrightarrow{Id} & \widehat{B} & \longrightarrow & B' & \longrightarrow & B & \longleftarrow & B'' & \longleftarrow & \widetilde{B} & \xleftarrow{Id} & \widetilde{B} & \longrightarrow & \mathbf{R}^3 Z \\
& \nwarrow Id & \uparrow & & \uparrow m' & & \uparrow r' & & \uparrow r & & \uparrow r'' & & \uparrow k' & & \uparrow & & \nearrow Id \\
& & \mathbf{R}^3 X & \xrightarrow{\mathbf{R}^2 \mu_X} & \mathbf{R}^4 X & \xrightarrow{\mathbf{R}^3 f'} & \mathbf{R}^3 S & \xrightarrow{\mathbf{R}^2 I_S} & \mathbf{R}^2 cyl(w', g') & \xleftarrow{\mathbf{R}^2 I_V} & \mathbf{R}^3 V & \xleftarrow{\mathbf{R}^3 v'} & \mathbf{R}^4 Z & \xleftarrow{\mathbf{R}^2 \mu_Z} & \mathbf{R}^3 Z
\end{array}$$

where the left triangle consists of

$$\begin{array}{c}
\mathbf{R}^3 X \\
\swarrow Id \quad \uparrow \mathbf{R}^2 \lambda_{\mathbf{R}X} \quad \nwarrow Id \\
\mathbf{R}^4 X \\
\uparrow \tau \\
\mathbf{R}^3 X \xrightarrow{\mathbf{R}^2 \mu_X} \mathbf{R}^4 X \xrightarrow{m'} \widehat{B} \xleftarrow{m} \mathbf{R}^4 X \xleftarrow{\mathbf{R}^2 \mu_X} \mathbf{R}^3 X .
\end{array}$$

whereas the right triangle is constructed analogously. Then by 3.1.6 we have that  $G \circ F \sim G' \circ F'$ , that finishes the proof.  $\square$

**LEMMA 3.1.9.** *The zig-zag  $F$  given by*

$$X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f} T \xleftarrow{w} \mathbf{R}Y \xrightarrow{\lambda_Y} Y$$

*is related to the following zig-zag, that will be denote by  $\mathbf{R}F$*

$$X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2 X \xrightarrow{\mathbf{R}f} \mathbf{R}T \xleftarrow{\mathbf{R}w} \mathbf{R}^2 Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y.$$

*Proof.* It is enough to consider the small hammock  $\mathcal{H}^0$

$$\begin{array}{ccccccccccc}
& & \mathbf{R}X & \xrightarrow{\mu_X} & \mathbf{R}^2 X & \xrightarrow{\mathbf{R}f} & \mathbf{R}T & \xleftarrow{\mathbf{R}w} & \mathbf{R}^2 Y & \xleftarrow{\mu_Y} & \mathbf{R}Y \\
& \swarrow Id & \downarrow Id & & \downarrow \lambda_{\mathbf{R}X} & & \downarrow \lambda_T & & \downarrow \lambda_{\mathbf{R}Y} & & \downarrow Id \\
\mathbf{R}X & \xleftarrow{\quad} & \mathbf{R}X & \xrightarrow{Id} & \mathbf{R}X & \xrightarrow{f} & T & \xleftarrow{w} & \mathbf{R}Y & \xleftarrow{Id} & \mathbf{R}Y \\
& \nwarrow Id & & & & & & & & & \nearrow Id
\end{array}$$

that is a particular case of a usual hammock. Then the statement follows from 3.1.6.  $\square$

**LEMMA 3.1.10.** *Given an element  $F$  of  $\mathbf{T}(X, Y)$  given by*

$$X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f} T \xleftarrow{w} \mathbf{R}Y \xrightarrow{\lambda_Y} Y$$

and an equivalence  $s : RX \rightarrow S$ , then  $F$  is related to the following zig-zag, that will be denote by  $F_S$

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rs} RS \xrightarrow{I_S} cyl(s, f) \xleftarrow{I_T} RT \xleftarrow{Rw} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y .$$

*Proof.* By 2.4.9 we deduce the existence of  $H : cyl(RX) \rightarrow cyl(s, f)$  satisfying the following property. If  $I_{RX}, J_{RX}$  are the canonical inclusions of  $R^2X$  into  $cyl(RX)$  then  $H \circ I_{RX} = I_S \circ Rs$  and  $H \circ J_{RX} = I_T \circ Rf$ .

Hence, we have the commutative diagram

$$\begin{array}{ccccc} R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y \\ \downarrow J_{RX} & & \downarrow I_T & & \downarrow Id \\ cyl(RX) & \xrightarrow{H} & cyl(s, f) & \xleftarrow{I_T \circ Rw} & R^2Y \\ \uparrow I_{RX} & & \uparrow Id & & \uparrow Id \\ R^2X & \xrightarrow{Rs} & RS & \xrightarrow{I_S} & cyl(s, f) \xleftarrow{I_T} RT \xleftarrow{Rw} R^2Y \end{array}$$

where all the vertical arrows are equivalences by the properties of  $cyl$ .

If  $\rho : cyl(RX) \rightarrow R^2X$  is such that  $\rho \circ J_{RX} = \rho \circ J_{RX} = Id$ , then the above diagram can be completed to

$$\begin{array}{ccccccccccc} & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y & \xleftarrow{\mu_Y} & RY \\ & \swarrow Id & \downarrow J_{RX \circ \mu_X} & \downarrow J_{RX} & \downarrow I_T & \downarrow Id & \downarrow Id & \downarrow Id & \downarrow Id & \downarrow \mu_Y & \searrow Id \\ RX & \xleftarrow{\lambda_{RX} \circ \rho} & cyl(RX) & \xrightarrow{Id} & cyl(RX) & \xrightarrow{H} & cyl(s, f) & \xleftarrow{I_T \circ Rw} & R^2Y & \xleftarrow{Id} & R^2Y \xrightarrow{\lambda_{RY}} RY, \\ & \swarrow Id & \uparrow I_{RX \circ \mu_X} & \uparrow I_{RX} & \uparrow Id & \uparrow Id & \uparrow Id & \uparrow Id & \uparrow Id & \uparrow \mu_Y & \searrow Id \\ & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{I_S \circ Rs} & cyl(s, f) & \xleftarrow{I_T \circ Rw} & R^2Y & \xleftarrow{\mu_Y} & RY \end{array}$$

that implies that  $F_S$  is related to the zig-zag  $RF$  given in 3.1.9. Then the result follows from the transitivity of  $\sim$ .  $\square$

The next lemma can be proved in a similar way.

**LEMMA 3.1.11.** *Consider the element  $F$  of  $T(X, Y)$*

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y$$

and an equivalence  $u : RY \rightarrow U$ . Then  $F$  is related to the zig-zag  $F^u$  given by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RT \xrightarrow{I_T} cyl(w, u) \xleftarrow{I_U} RU \xleftarrow{Ru} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y .$$

**LEMMA 3.1.12.** *The composition of morphisms in  $Ho\mathcal{D}$  allows to “delete pairs of inverse arrows”. In other words, assume given composable zig-zags  $F$  and  $G$  described respectively as*

$$\begin{array}{ccccccccc} X & \xleftarrow{\lambda_X} & RX & \xrightarrow{f} & U & \xleftarrow{u} & RS & \xleftarrow{s} & RY & \xrightarrow{\lambda_Y} & Y \\ Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{s} & RS & \xrightarrow{g} & V & \xleftarrow{v} & RZ & \xrightarrow{\lambda_Z} & Z . \end{array}$$

*Then the zig-zag  $G \circ F$  is equivalent to the composition of the zig-zags  $F'$  and  $G'$  given by*

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f} U \xleftarrow{u} RS \xrightarrow{\lambda_S} S \quad ; \quad S \xleftarrow{\lambda_S} RS \xrightarrow{g} V \xleftarrow{v} RZ \xrightarrow{\lambda_Z} Z .$$

*Proof.* Since the following diagram commutes

$$\begin{array}{ccccc} U & \xleftarrow{s \circ u} & RY & \xrightarrow{g \circ s} & V \\ Id \downarrow & & s \downarrow & & Id \downarrow \\ U & \xleftarrow{u} & RS & \xrightarrow{g} & V \end{array}$$

we deduce from the functoriality of  $cyl$  a morphism  $\alpha : cyl(s \circ u, g \circ s) \rightarrow cyl(u, g)$  such that the diagram

$$\begin{array}{ccccc} RU & \xrightarrow{I_U} & cyl(s \circ u, g \circ s) & \xleftarrow{I_V} & RV \\ Id \downarrow & & \alpha \downarrow & & Id \downarrow \\ RU & \xrightarrow{J_U} & cyl(u, g) & \xleftarrow{J_V} & RV \end{array}$$

commutes. In addition, by 2.2.6,  $\alpha \in E$ . Then the above diagram can be completed to the following small hammock

$$\begin{array}{ccccccccccccccc} & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RU & \xrightarrow{I_U} & cyl(s \circ u, g \circ s) & \xleftarrow{I_V} & RV & \xleftarrow{Rv} & R^2Z & \xleftarrow{\mu_X} & RZ & & \\ & Id \swarrow & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow \alpha & & \downarrow Id & & \downarrow Id & & \downarrow Id & & Id \searrow \\ RX & & & & & & & & & & & & & & & & RZ \\ & Id \swarrow & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & Id \searrow \\ & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RU & \xrightarrow{J_U} & cyl(u, g) & \xleftarrow{J_V} & RV & \xleftarrow{Rv} & R^2Z & \xleftarrow{\mu_X} & RZ & & \end{array}$$

that relates  $G \circ F$  to  $G' \circ F'$ . □

**LEMMA 3.1.13.** *The composition of the following zig-zags  $F$  and  $G$*

$$\begin{array}{ccccccc} X & \xleftarrow{\lambda_X} & RX & \xrightarrow{f} & U & \xleftarrow{u} & RY & \xrightarrow{\lambda_Y} & Y \\ Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{u} & U & \xrightarrow{g} & V & \xleftarrow{v} & RZ & \xrightarrow{\lambda_Z} & Z \end{array}$$

*is equivalent to*

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f} U \xrightarrow{g} V \xleftarrow{v} RZ \xrightarrow{\lambda_Z} Z .$$



*Proof.* By definition  $G \circ F$  is

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RU \xrightarrow{I_U} cyl(u, g \circ u) \xleftarrow{I_V} RV \xleftarrow{Rv} R^2Z \xleftarrow{\mu_X} RZ \xrightarrow{\lambda_X} X.$$

Since  $g \circ u = Id_{V \circ (g \circ u)}$ , from 2.2.9 we deduce the existence of  $\beta : cyl(u, g \circ u) \rightarrow RV$  such that  $\beta \circ I_U = Rg$  and  $\beta \circ I_V = Id_{RV}$ . Then,  $\beta \in E$  and we have the hammock

$$\begin{array}{ccccccccccccccc} & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RU & \xrightarrow{I_U} & cyl(u, g \circ u) & \xleftarrow{I_V} & RV & \xleftarrow{Rv} & R^2Z & \xleftarrow{\mu_X} & RZ & & \\ & \swarrow Id & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow \beta & & \downarrow Id & & \downarrow Id & & \downarrow Id & \searrow Id & \\ RX & & & & & & & & & & & & & & & & RZ \\ & \swarrow Id & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & \searrow Id & \\ & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RU & \xrightarrow{Rg} & RV & \xleftarrow{Id} & RV & \xleftarrow{Rv} & R^2Z & \xleftarrow{\mu_X} & RZ & & \end{array}$$

To finish the proof, it suffices to take into account 3.1.9.  $\square$

**Proof of 3.1.5.**

To see that  $Ho\mathcal{D}$  is a category, it remains to check that the composition of morphisms is associative and that it has a unit.

**Unit:** given  $X$  in  $\mathcal{D}$ , the unit for the composition in  $\text{Hom}_{Ho\mathcal{D}}(X, X)$  is  $1_X = \gamma(Id_X)$ , that is the morphism represented by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{Id} RX \xleftarrow{Id} RX \xrightarrow{\lambda_X} X.$$

Indeed, if  $\hat{f} \in \text{Hom}_{Ho\mathcal{D}}(X, Y)$  is the class of  $X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y$  then by definition  $\hat{f} \cdot 1_X$  is represented by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{I_{RX}} cyl(Id_{RX}, f) \xleftarrow{I_T} RT \xleftarrow{Rw} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y.$$

From the commutative diagram

$$\begin{array}{ccc} RX & \xrightarrow{Id} & RX \\ f \downarrow & & \downarrow f \\ T & \xrightarrow{Id} & T \end{array}$$

we deduce by 2.2.9 the existence of an equivalence  $\alpha : cyl(RX, f) \rightarrow RT$  such that  $\alpha \circ I_{RX} = Rf$  and  $\alpha \circ I_T = Id_T$ , so we have the hammock

$$\begin{array}{ccccccc} & & R^2X & \xrightarrow{R\mu_X} & R^3X & \xrightarrow{RI_{RX}} & Rcyl(Id, f) & \xleftarrow{RI_T \circ R^2w \circ R\mu_Y} & R^2Y & & \\ & \swarrow Id & \downarrow R\mu_X & & \downarrow R\alpha & & \downarrow R\mu_Y & & \downarrow Id & & \\ R^2X & \xleftarrow{R\lambda_{RX}} & R^3X & \xrightarrow{R^2f} & R^2T & \xleftarrow{R^2w} & R^3Y & \xrightarrow{R\lambda_{RX}} & R^2Y. & & \\ & \swarrow Id & \downarrow \mu_{RX} & & \downarrow \mu_T & & \downarrow \mu_{RY} & & \downarrow Id & & \\ & & R^2X & \xrightarrow{Rf} & RT & \xleftarrow{Rw} & R^2Y & & & & \end{array}$$

Therefore,  $\widehat{f} \circ 1_X = \widehat{f}$ , and it can be proved analogously that  $1_Y \circ \widehat{f} = \widehat{f}$ .

**Associativity:** Let  $\widehat{f}$ ,  $\widehat{g}$  and  $\widehat{h}$  three composable morphisms in  $Ho\mathcal{D}$ , represented respectively by the following zig-zags  $F$ ,  $G$  and  $H$

$$\begin{array}{ccccccc} X & \xleftarrow{\lambda_X} & RX & \xrightarrow{f} & U & \xleftarrow{u} & RY & \xrightarrow{\lambda_Y} & Y \\ Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{g} & V & \xleftarrow{v} & RZ & \xrightarrow{\lambda_Z} & Z \\ Z & \xleftarrow{\lambda_Z} & RZ & \xrightarrow{h} & W & \xleftarrow{w} & RS & \xrightarrow{\lambda_S} & S \end{array}$$

Let us see that  $(\widehat{h} \circ \widehat{f}) \circ \widehat{g} = \widehat{h} \circ (\widehat{g} \circ \widehat{f})$ . Consider the diagram

$$\begin{array}{ccccccccc} R^2U & \xleftarrow{R^2u} & R^3Y & \xrightarrow{R^2g} & R^2V & \xleftarrow{R^2v} & R^3Z & \xrightarrow{R^2h} & R^2W \\ & \searrow RI_U & & \searrow RI_V & & \searrow RJ_V & & \searrow RI_W & \\ & & R\text{cyl}(u, g) & & & & R\text{cyl}(v, h) & & \\ & & \searrow I & & \searrow J & & & & \\ & & & & \text{cyl}(I_V, J_V) & & & & \end{array}$$

where  $I_V$ ,  $I_W$  and  $J$  are equivalences.

Let us check that  $(\widehat{h} \circ \widehat{g}) \circ \widehat{f}$  and  $\widehat{h} \circ (\widehat{g} \circ \widehat{f})$  coincides with the morphism represented by the zig-zag  $L$  given by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\hat{\mu}_X} R^3X \xrightarrow{R^2f} R^2U \xrightarrow{I \circ RI_U} \text{cyl}(I_V, J_V) \xleftarrow{J \circ RI_W} R^2W \xleftarrow{R^2w} R^3S \xleftarrow{\hat{\mu}_S} RS \xrightarrow{\lambda_S} S,$$

where  $\hat{\mu} : R \rightarrow R^3$  is  $\hat{\mu}_M = R\mu_M \circ \mu_M$ , for every object  $M$  in  $\mathcal{D}$ .

Having into account the equivalences  $u : RY \rightarrow U$  and  $v : RZ \rightarrow V$ , by 3.1.10 it follows that  $G \sim G_u$  and  $H \sim H_v$ , where  $G_u$  and  $H_v$  are respectively

$$\begin{array}{ccccccccccc} Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\mu_Y} & R^2Y & \xrightarrow{Ru} & RU & \xrightarrow{I_U} & \text{cyl}(u, g) & \xleftarrow{I_V} & RV & \xleftarrow{Rv} & R^2Z & \xleftarrow{\mu_Z} & RZ & \xrightarrow{\lambda_Z} & Z \\ Z & \xleftarrow{\lambda_Z} & RZ & \xrightarrow{\mu_Z} & R^2Z & \xrightarrow{Rv} & RV & \xrightarrow{J_V} & \text{cyl}(v, h) & \xleftarrow{I_W} & RW & \xleftarrow{Rw} & R^2S & \xleftarrow{\mu_S} & RS & \xrightarrow{\lambda_S} & S. \end{array}$$

Therefore  $\widehat{h} \circ \widehat{g}$  is the class of  $H_v \circ G_u$ , that by 3.1.12 coincides with the composition of zig-zags

$$\begin{array}{ccccccc} Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\mu_Y} & R^2Y & \xrightarrow{Ru} & RU & \xrightarrow{I_U} & \text{cyl}(u, g) & \xleftarrow{I_V} & RV & \xrightarrow{\lambda_V} & V \\ V & \xleftarrow{\lambda_V} & RV & \xrightarrow{J_V} & \text{cyl}(v, h) & \xleftarrow{I_W} & RW & \xleftarrow{Rw} & R^2S & \xleftarrow{\mu_S} & RS & \xrightarrow{\lambda_S} & S, \end{array}$$

and this is by definition the zig-zag  $(H \circ G)'$

$$Y \xleftarrow{\lambda_Y} RY \xrightarrow{\hat{\mu}_Y} R^3Y \xrightarrow{R^2u} R^2U \xrightarrow{I \circ RI_U} \text{cyl}(I_V, J_V) \xleftarrow{J \circ RI_W} R^2S \xleftarrow{\mu_W} RS \xrightarrow{\lambda_S} S$$

where  $t' = I_W \circ R w \circ \mu_S : RS \rightarrow cyl(v, h)$ .

In addition,  $F$  is related to the zig-zag  $R^2F$  consisting of

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\hat{\mu}_X} R^3X \xrightarrow{R^2f} R^2U \xleftarrow{R^2u} R^3Y \xleftarrow{\hat{\mu}_Y} RY \xrightarrow{\lambda_Y} Y$$

and by 3.1.13 it follows that  $(H \circ G)' \circ R^2F \sim L$ .

On the other hand,  $G \circ F \sim G_u \circ RF$ , that becomes after deleting arrows in

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RU \xrightarrow{I_U} cyl(u, g) \xleftarrow{I_V} RV \xleftarrow{Rv} R^2Z \xleftarrow{\mu_Z} RZ \xrightarrow{\lambda_Z} Z$$

Hence the result of composing with  $H_v$  is related to the composition of

$$\begin{aligned} X &\xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RU \xrightarrow{I_U} cyl(u, g) \xleftarrow{I_V} RV \xrightarrow{\lambda_V} V \\ V &\xleftarrow{\lambda_V} RV \xrightarrow{J_V} cyl(v, h) \xleftarrow{I_W} RW \xleftarrow{Rw} R^2S \xleftarrow{\mu_S} RS \xrightarrow{\lambda_S} S, \end{aligned}$$

that is  $L$  by definition.

**Functoriality of  $\gamma : \mathcal{D} \rightarrow Ho\mathcal{D}$ :** we have by definition that the image under  $\gamma$  of an identity morphism in  $\mathcal{D}$  is an identity morphism in  $Ho\mathcal{D}$ . Given composable morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{D}$ , let us check that  $\gamma(g \circ f) = \gamma(g) \circ \gamma(f)$ .

By definition,  $\gamma(g \circ f)$  is represented by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{Rf} RY \xrightarrow{Rg} RZ \xleftarrow{Id} RZ \xrightarrow{\lambda_Z} Z$$

In the same way, the composition of the zig-zags

$$X \xleftarrow{\lambda_X} RX \xrightarrow{Rf} RY \xleftarrow{Id} RY \xrightarrow{\lambda_Y} Y \quad ; \quad Y \xleftarrow{\lambda_Y} RY \xrightarrow{Rg} RZ \xleftarrow{Id} RZ \xrightarrow{\lambda_Z} Z$$

is  $X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{R^2f} R^2Y \xrightarrow{I_{RY}} cyl(Id, Rg) \xleftarrow{I_{RZ}} R^2Z \xleftarrow{\mu_Z} RZ \xrightarrow{\lambda_Z} Z$  by definition. It follows from 2.2.9 the existence of  $\rho : cyl(Id, Rg) \rightarrow R^2Z$  such that  $\rho \circ I_{RY} = R^2g$  and  $\rho \circ I_{RZ} = Id_{R^2Z}$ . Hence,  $\rho$  is an equivalence and we have the following small hammock

$$\begin{array}{ccccccccccc} & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{R^2f} & R^2Y & \xrightarrow{I_{RY}} & cyl(Id, Rg) & \xleftarrow{I_{RZ}} & R^2Z & \xleftarrow{\mu_Z} & RZ & & \\ & Id \swarrow & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow \rho & & \downarrow Id & & \downarrow Id & Id \searrow \\ RX & & & & & & & & & & & & & RZ \\ & Id \swarrow & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow Id & Id \searrow \\ & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{R^2f} & R^2Y & \xrightarrow{R^2g} & R^2Z & \xleftarrow{Id} & R^2Z & \xleftarrow{\mu_Z} & RZ & \end{array}$$

where the lower zig-zag represents the morphism  $\gamma(g \circ f)$  by 3.1.9, and then the required equality holds.

**Universal property :** First,  $\gamma : \mathcal{D} \rightarrow Ho\mathcal{D}$  is such that  $\gamma(E) \subseteq \{\text{isomorphisms of } Ho\mathcal{D}\}$ . Indeed, if  $w : X \rightarrow Y$  is an equivalence, then  $Rw$  is so, and  $\gamma(w)^{-1}$  is the morphism in  $Ho\mathcal{D}$  given by

$$Y \xleftarrow{\lambda_Y} RY \xrightarrow{Id} RY \xleftarrow{Rw} RX \xrightarrow{\lambda_X} X$$

that clearly is the inverse of  $\gamma(w)$  (by 3.1.13).

It remains to see that the pair  $(Ho\mathcal{D}, \gamma)$  satisfies the universal property of the localized category  $\mathcal{D}[E^{-1}]$ .

Let  $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{C}$  be a functor such that  $\mathcal{F}$  maps equivalences into isomorphisms. We must prove that there exists an unique functor  $\mathcal{G} : Ho\mathcal{D} \rightarrow \mathcal{C}$  such that  $\mathcal{G} \circ \gamma = \mathcal{F}$ .

*Existence:* define  $\mathcal{G}$  as  $\mathcal{G}X = \mathcal{F}X$  if  $X$  is any object of  $\mathcal{D}$ . The image under  $\mathcal{G}$  of a morphism  $\hat{f}$  in  $Ho\mathcal{D}$  given by  $X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y$  is

$$\mathcal{G}(\hat{f}) = (\mathcal{F}\lambda_Y) \circ (\mathcal{F}w)^{-1} \circ (\mathcal{F}f) \circ (\mathcal{F}\lambda_X)^{-1} .$$

Note that  $\mathcal{G}(\hat{f})$  does not depend on the zig-zag chosen. Indeed, if two zig-zags  $L$  and  $L'$  represent  $\hat{f}$ , both will be related by a hammock as in (3.3). This hammock becomes, after applying  $\mathcal{F}$ , a commutative diagram in  $\mathcal{C}$ . As the equivalences are now isomorphisms, it follows that  $\mathcal{F}(L) = \mathcal{F}(L')$ .

In addition, the equality  $\mathcal{G} \circ \gamma = \mathcal{F}$  holds. To see this, let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{D}$ . By definition  $\mathcal{G}(\gamma f)$  is

$$\mathcal{F}\lambda_Y \circ (\mathcal{F}Id_{RY})^{-1} \circ \mathcal{F}(Rf) \circ (\mathcal{F}\lambda_X)^{-1} = \mathcal{F}\lambda_Y \circ \mathcal{F}(Rf) \circ (\mathcal{F}\lambda_X)^{-1} ,$$

that agrees with  $\mathcal{F}(f)$  since  $\lambda_Y \circ Rf = f \circ \lambda_X$  in  $\mathcal{D}$ .

Next we check that  $\mathcal{G}$  is functorial. It is clear that  $\mathcal{G}$  maps identities into identities, since  $\mathcal{G}(Id_X) = \mathcal{G}(\gamma(Id_X)) = Id_{\mathcal{F}(X)}$ .

On the other hand, let  $\hat{f}$  and  $\hat{g}$  be composable morphisms in  $Ho\mathcal{D}$  represented by the respective zig-zags

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y \quad ; \quad Y \xleftarrow{\lambda_Y} RY \xrightarrow{g} S \xleftarrow{v} RZ \xrightarrow{\lambda_Z} Z.$$

We must see that  $\mathcal{G}(\hat{g}) \circ \mathcal{G}(\hat{f}) = \mathcal{G}(\hat{g} \circ \hat{f})$ , that is, the following composition of morphisms must coincide in  $\mathcal{C}$

$$\begin{aligned} \mathcal{F}(RX) &\xrightarrow{\mathcal{F}f} \mathcal{F}T \xrightarrow{(\mathcal{F}w)^{-1}} \mathcal{F}RY \xrightarrow{\mathcal{F}g} \mathcal{F}S \xrightarrow{(\mathcal{F}v)^{-1}} \mathcal{F}(RZ) \\ \mathcal{F}(RX) &\xrightarrow{\mathcal{F}\mu_X} \mathcal{F}(R^2X) \xrightarrow{\mathcal{F}(Rf)} \mathcal{F}(RT) \xrightarrow{\mathcal{F}Id_T} \mathcal{F}cyl(w, g) \xrightarrow{(\mathcal{F}Is)^{-1}} \mathcal{F}RS \xrightarrow{(\mathcal{F}Rv)^{-1}} \mathcal{F}(R^2Z) \xrightarrow{(\mathcal{F}\mu_Z)^{-1}} \mathcal{F}(RZ). \end{aligned}$$

where we have already deleted in  $\mathcal{G}(\hat{g}) \circ \mathcal{G}(\hat{f})$  and  $\mathcal{G}(\hat{g} \circ \hat{f})$  the isomorphisms  $(\mathcal{F}\lambda_X)^{-1}$  and  $\mathcal{F}\lambda_Z$ . Then, it suffices to prove the commutativity in  $\mathcal{C}$  of the following diagrams (I), (II) and (III)

$$\begin{array}{ccccccc}
\mathcal{F}(\mathbf{R}X) & \xrightarrow{\mathcal{F}f} & \mathcal{F}T & \xrightarrow{(\mathcal{F}w)^{-1}} & \mathcal{F}RY & \xrightarrow{\mathcal{F}g} & \mathcal{F}S \xrightarrow{(\mathcal{F}v)^{-1}} \mathcal{F}(\mathbf{R}Z) \\
\mathcal{F}\mu_X \downarrow & & \uparrow \mathcal{F}\lambda_T & & & \uparrow \mathcal{F}\lambda_S & \downarrow \mathcal{F}\mu_Z \\
\mathcal{F}(\mathbf{R}^2X) & \xrightarrow{\mathcal{F}(\mathbf{R})f} & \mathcal{F}(\mathbf{R}T) & \xrightarrow{\mathcal{F}I_T} & \mathcal{F}cyl(w, g) & \xrightarrow{(\mathcal{F}I_S)^{-1}} & \mathcal{F}\mathbf{R}S \xrightarrow{(\mathcal{F}(\mathbf{R}v))^{-1}} \mathcal{F}(\mathbf{R}^2Z) .
\end{array}
\quad \begin{array}{c} (I) \qquad (II) \qquad (III) \end{array}$$

To see (I) and (III), just note that in  $\mathcal{D}$  we have the commutativity of

$$\begin{array}{ccc}
\mathbf{R}X & \xrightarrow{f} & T \\
\mu_X \downarrow & & \uparrow \lambda_T \\
\mathbf{R}^2X & \xrightarrow{\mathbf{R}f} & \mathbf{R}T
\end{array}
\quad ; \quad
\begin{array}{ccc}
\mathbf{R}Z & \xrightarrow{v} & S \\
\mu_Z \downarrow & & \uparrow \lambda_S \\
\mathbf{R}^2Z & \xrightarrow{\mathbf{R}v} & \mathbf{R}S
\end{array}$$

Indeed, the statement follows from the equalities  $\lambda_T \circ \mathbf{R}f = f \circ \lambda_{\mathbf{R}X}$  and  $\lambda_{\mathbf{R}X} \circ \mu_X = Id_{\mathbf{R}X}$ , and analogously for the right diagram.

The commutativity of (II) follows from the commutativity of the diagrams below in  $\mathcal{D}$  and  $\mathcal{C}$  respectively

$$\begin{array}{ccc}
T & \xleftarrow{w} & \mathbf{R}Y \xrightarrow{g} S \\
\uparrow \lambda_T & & \uparrow \lambda_{\mathbf{R}Y} \quad \uparrow \lambda_S \\
\mathbf{R}T & \xleftarrow{\mathbf{R}w} & \mathbf{R}^2Y \xrightarrow{\mathbf{R}g} \mathbf{R}S
\end{array}
\quad ; \quad
\begin{array}{ccc}
\mathcal{F}(\mathbf{R}T) & \xleftarrow{\mathcal{F}(\mathbf{R}w)} & \mathcal{F}(\mathbf{R}^2Y) \xrightarrow{\mathcal{F}(\mathbf{R}g)} \mathcal{F}(\mathbf{R}S) \\
& \searrow \mathcal{F}I_T & \swarrow \mathcal{F}I_S \\
& & \mathcal{F}cyl(w, g)
\end{array}$$

Let us see that  $\mathcal{F}I_T \circ \mathcal{F}(\mathbf{R}w) = \mathcal{F}I_S \circ \mathcal{F}(\mathbf{R}g)$  in  $\mathcal{C}$ . From the property 2.2.9 of  $cyl$  we deduce the following diagram involving  $cyl(\mathbf{R}Y) = cyl(Id_{\mathbf{R}Y}, Id_{\mathbf{R}Y})$

$$\begin{array}{ccc}
\mathbf{R}^2Y & \xrightarrow{\quad Id \quad} & \\
\downarrow I_{\mathbf{R}Y} & & \uparrow J_{\mathbf{R}Y} \\
& cyl(\mathbf{R}Y) \xrightarrow{\quad \rho \quad} \mathbf{R}^2Y & \\
\uparrow J_{\mathbf{R}Y} & & \downarrow I_{\mathbf{R}Y} \\
\mathbf{R}^2Y & \xrightarrow{\quad Id \quad} &
\end{array}$$

Hence,  $\rho \in E$  and since  $\mathcal{F}\rho \circ \mathcal{F}I_{\mathbf{R}Y} = \mathcal{F}\rho \circ \mathcal{F}J_{\mathbf{R}Y} = Id_{\mathbf{R}^2Y}$  then  $\mathcal{F}I_{\mathbf{R}Y} = \mathcal{F}J_{\mathbf{R}Y}$  in  $\mathcal{C}$ . On the other hand, by 2.4.9 we have a morphism  $H : cyl(\mathbf{R}Y) \rightarrow cyl(w, g)$  such that  $H \circ J_{\mathbf{R}Y} = I_S \circ \mathbf{R}g$  and  $H \circ I_{\mathbf{R}Y} = I_T \circ \mathbf{R}w$ . Applying  $\mathcal{F}$  we deduce that  $\mathcal{F}(I_T \circ \mathbf{R}w) = \mathcal{F}(I_S \circ \mathbf{R}g)$ .

*Uniqueness:* Assume that  $\mathcal{G}' : Ho\mathcal{D} \rightarrow \mathcal{C}$  is such that  $\mathcal{G}' \circ \gamma = \mathcal{F}$ . The equality  $\mathcal{G}' = \mathcal{G}$  is deduced from  $\mathcal{G}' \circ \gamma = \mathcal{G} \circ \gamma$ , together with the following lemma.  $\square$

**LEMMA 3.1.14.** *The morphism  $\hat{f}$  in  $Ho\mathcal{D}$  given by  $X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y$  is equal to  $\gamma(\lambda_Y) \circ (\gamma(w))^{-1} \circ \gamma(f) \circ (\gamma(\lambda_X))^{-1}$ .*

*Proof.* Firstly, note that if  $S$  is any object in  $\mathcal{D}$ , then  $\gamma(\lambda_S : RS \rightarrow S)$  is represented by  $RS \xleftarrow{\lambda_{RS}} R^2S \xrightarrow{\lambda_{RS}} RS \xleftarrow{Id} RS \xrightarrow{\lambda_S} S$ .

Indeed, by definition  $\gamma(\lambda_S)$  is the class of  $RS \xleftarrow{R\lambda_S} R^2S \xrightarrow{\lambda_{RS}} RS \xleftarrow{Id} RS \xrightarrow{\lambda_S} S$ , and from the naturality of  $\lambda$  it follows that  $\lambda_{RS} \circ \lambda_S = R\lambda_S \circ \lambda_S = \alpha$ . Therefore we have the following hammock relating both zig-zags

$$\begin{array}{ccccccc}
 & & R^2S & \xrightarrow{\lambda_{RS}} & RS & \xleftarrow{Id} & RS \\
 & Id \swarrow & \downarrow Id & & \downarrow \lambda_S & & \downarrow Id \\
 R^2S & \xleftarrow{Id} & R^2S & \xrightarrow{\alpha} & S & \xleftarrow{\lambda_S} & RS \\
 & \swarrow Id & \uparrow Id & & \uparrow \lambda_S & & \uparrow Id \\
 & & R^2S & \xrightarrow{R\lambda_S} & RS & \xleftarrow{Id} & RS
 \end{array}$$

Hence,  $\gamma(f) \circ (\gamma(\lambda_X))^{-1}$  is represented by the composition of the zig-zags

$$X \xleftarrow{\lambda_X} RX \xrightarrow{Id} RX \xleftarrow{\lambda_{RX}} R^2X \xrightarrow{\lambda_{RX}} RX \quad ; \quad RX \xleftarrow{\lambda_{RX}} R^2X \xrightarrow{Rf} RT \xleftarrow{Id} RT \xrightarrow{\lambda_T} T$$

that is by definition  $X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{I_{RX}} cyl(\lambda_{RX}, Rf) \xleftarrow{I_{RT}} R^2T \xleftarrow{\mu_T} RT \xrightarrow{\lambda_T} T$ .

From the equality  $\lambda_T \circ Rf = g \circ \lambda_{RX}$  and the property 2.2.9 of  $cyl$  we get the small hammock

$$\begin{array}{ccccccc}
 & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{I_{RX}} & cyl(\lambda_{RX}, Rf) \\
 & Id \swarrow & \downarrow Id & & \downarrow Id & & \downarrow \\
 RX & \xleftarrow{Id} & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RT \\
 & \swarrow Id & & & & & \downarrow \lambda_T \\
 & & & & & & RT \\
 & & & & & & \downarrow Id \\
 & & & & & & RT
 \end{array}$$

Then  $\gamma(f) \circ (\gamma(\lambda_X))^{-1}$  is represented by  $X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RT \xleftarrow{Id} RT \xrightarrow{\lambda_T} T$ .

Compose with  $T \xleftarrow{\lambda_T} RT \xrightarrow{Id} RT \xleftarrow{Rw} R^2Y \xrightarrow{\lambda_{RY}} RY$  (that represents the morphism  $[\gamma(w)]^{-1}$ ) and delete arrows in a suitable way to obtain that  $[\gamma(w)]^{-1} \circ \gamma(f) \circ (\gamma(\lambda_X))^{-1}$  is given by  $X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf} RT \xleftarrow{Rw} R^2Y \xrightarrow{\lambda_{RY}} RY$ .

To finish it suffices to compose this zig-zag with  $RY \xleftarrow{\lambda_{RY}} R^2Y \xrightarrow{\lambda_{RY}} RY \xleftarrow{Id} RY \xrightarrow{\lambda_Y} Y$ .

Again  $\lambda_T \circ Rw = w \circ \lambda_{RY}$ , and as before we get

$$\begin{array}{ccccccccccc}
 & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{R\mu_X} & R^3X & \xrightarrow{Rf} & R^2T & \xrightarrow{I_{RX}} & cyl(Rw, \lambda_{RY}) \\
 & Id \swarrow & \downarrow Id & & \downarrow \lambda_{RX} & & \downarrow \lambda_{R^2X} & & \downarrow \lambda_T & & \downarrow \\
 RX & \xleftarrow{Id} & RX & \xrightarrow{Id} & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rf} & RT & \xrightarrow{Id} & RT \\
 & \swarrow Id & & & & & & & & & \downarrow \\
 & & & & & & & & & & RT \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & RT \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & R^2Y \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & RY
 \end{array}$$

Hence  $\gamma(\lambda_Y) \circ [\gamma(w)]^{-1} \circ \gamma(f) \circ (\gamma(\lambda_X))^{-1}$  is the class of the hammock induced by the lower row of the above hammock, and from 3.1.9 the required equality follows.  $\square$

**COROLLARY 3.1.15.** *A final object in  $\mathcal{D}$  is also a final object in  $Ho\mathcal{D}$ .*

*Proof.* Indeed, if  $X \xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} R1 \xrightarrow{\lambda_1} 1$  represents the morphism  $\hat{f} : X \rightarrow 1$  in  $Ho\mathcal{D}$ , then  $\hat{f} = \gamma(\rho)$ , where  $\rho : X \rightarrow 1$  is the trivial morphism in  $\mathcal{D}$ . To see this, just consider the hammock

$$\begin{array}{ccccccc}
 & & RX & \xrightarrow{R\rho} & R1 & \xleftarrow{Rw} & R1 \\
 & \swarrow Id & \downarrow Id & & \downarrow \lambda_1 & \downarrow Id & \searrow Id \\
 RX & \xleftarrow{Id} & RX & \longrightarrow & 1 & \xleftarrow{\lambda_S} & R1 \xrightarrow{Id} R1 . \\
 & \swarrow Id & \uparrow Id & & \uparrow \lambda_1 & \uparrow Id & \nearrow Id \\
 & & RX & \xrightarrow{Rf} & R1 & \xleftarrow{Id} & R1
 \end{array}$$

$\square$

**COROLLARY 3.1.16.** *Assume that the trivial morphism  $\sigma_0 : 0 \rightarrow R0$  is an isomorphism in  $\mathcal{D}$ , where  $0$  is an initial object in  $\mathcal{D}$ . Then  $0$  is also an initial object in  $Ho\mathcal{D}$ .*

**REMARK 3.1.17.**

- i) Since  $\mathbf{s}$  commutes with coproducts up to equivalence, we deduce that  $\sigma_0$  is always an equivalence, and  $\lambda_0 \circ \sigma_0 = Id$  because  $0$  is initial.
- ii) In the examples considered in this work, the simple functor  $\mathbf{s}$  always commutes with coproducts (that is, the transformation  $\sigma$  of 2.1.4 is an isomorphism). In particular, the hypothesis in the previous corollary holds.

*Proof.* Let  $F : 0 \xleftarrow{\lambda_0} R0 \xrightarrow{f} T \xleftarrow{w} RX \xrightarrow{\lambda_X} X$  be a zig-zag representing the morphism  $\hat{f} : 0 \rightarrow X$  in  $Ho\mathcal{D}$ .

By assumption,  $R0$  is isomorphic to  $0$ , so  $R0$  is an initial object in  $\mathcal{D}$ . Then we have the following commutative diagram

$$\begin{array}{ccccc}
 R0 & \xrightarrow{f} & T & \xleftarrow{w} & RX \\
 \uparrow Id & & \uparrow w & & \uparrow Id \\
 R0 & \longrightarrow & RX & \xleftarrow{Id} & RX ,
 \end{array}$$

that gives rise to a hammock relating  $F$  to  $\gamma(0 \rightarrow X)$ .  $\square$

### 3.2 Descent categories with $\lambda$ quasi-invertible

In the description of  $Ho\mathcal{D}$  given in the previous section, the zig-zags that represent the morphisms in  $Ho\mathcal{D}$  consists of four arrows instead of two (that is the situation in the calculus of fractions case). The reason is that the cylinder of two morphisms  $A \xleftarrow{f} B \xrightarrow{g} C$  gives rise to  $RA \rightarrow cyl(f, g) \leftarrow RC$ , so we need to attach  $\lambda$ s in order to recover  $A$  and  $C$ .

If  $\lambda$  is “quasi-invertible” this problem disappears, and the description of the morphisms in  $Ho\mathcal{D}$  become easier.

Throughout this section,  $(\mathcal{D}, E, \mathbf{s}, \mu, \lambda)$  denotes a simplicial descent category.

**DEFINITION 3.2.1.** We will say that  $\lambda : R \rightarrow Id_{\mathcal{D}}$  is *quasi-invertible* if there exists a natural transformation  $\rho : Id_{\mathcal{D}} \rightarrow R$  such that  $\lambda \circ \rho : Id_{\mathcal{D}} \rightarrow Id_{\mathcal{D}}$  is the identity natural transformation. That is,  $\lambda_X \circ \rho_X = Id_X$  for every object  $X$  in  $\mathcal{D}$ .

An example of such situation is the category of chain complexes.

**PROPOSITION 3.2.2.** Assume that  $\lambda$  is quasi-invertible. Then the cylinder object of two morphisms  $A \xleftarrow{f} B \xrightarrow{g} C$  in  $\mathcal{D}$  has the following properties

1) there exists morphisms in  $\mathcal{D}$ , functorial in  $(f, g)$

$$j_A : A \rightarrow cyl(f, g) \quad j_B : B \rightarrow cyl(f, g)$$

such that  $j_A$  (resp.  $j_B$ ) is in  $E$  if and only if  $g$  (resp.  $f$ ) is so.

2) If  $f = g = Id_A$ , there exists an equivalence  $P : cyl(A) \rightarrow A$  in  $\mathcal{D}$  such that the composition of  $P$  with the inclusions  $j_A, j'_A : A \rightarrow cyl(A)$  given in 1) is equal to the identity  $A$ .

3) there exists  $H : cyl(A) \rightarrow cyl(f, g)$  such that  $H$  composed with  $j_A$  and  $j'_A$  is equal to  $j_A \circ f$  and  $j_B \circ g$  respectively.

*Proof.* As usual, 3) follows from 1).

To see 1),  $j_A$  and  $j_B$  are defined as the compositions

$$A \xrightarrow{\rho_A} RA \xrightarrow{I_A} cyl(f, g) \quad B \xrightarrow{\rho_B} RB \xrightarrow{I_B} cyl(f, g) \quad .$$

Since  $\lambda_A \circ \rho_A = Id_A$  and  $\lambda_A \in E$ , we deduce that  $\rho_A \in E$ . Hence, 1) follows from the properties of the functor  $cyl$ .

Finally, if  $\mathcal{D}$  is any descent category, there exists  $P' : cyl(A) \rightarrow RA$  such that  $P' \circ I_A = P' \circ J_A = Id_{RA}$ , where  $I_A, J_A$  denote the canonical inclusions of  $RA$



into  $\text{cyl}(A)$ .

Therefore,  $P = \lambda_A \circ P'$  satisfies 3) trivially.  $\square$

We can replace the maps  $I_A : \text{RA} \rightarrow \text{cyl}(f, g)$  (resp.  $I_C$ ) by  $j_A$  (resp.  $j_C$ ) in the proofs of the previous section. In this way we obtain the following proposition.

**PROPOSITION 3.2.3.** *If  $\mathcal{D}$  is a simplicial descent category with  $\lambda$  quasi-invertible, then the morphisms in  $\text{Ho}\mathcal{D}$  can be described as follows*

*Given objects  $X, Y$  in  $\mathcal{D}$ ,*

$$\text{Hom}_{\text{Ho}\mathcal{D}}(X, Y) = \text{T}'(X, Y) / \sim$$

*where an element  $F$  of  $\text{T}'(X, Y)$  is a zig-zag*

$$X \xrightarrow{f} T \xleftarrow{w} Y, \quad w \in \text{E}.$$

*If  $X \xrightarrow{g} S \xleftarrow{u} Y$  is another element  $G \in \text{T}'(X, Y)$ , then  $F \sim G$  if and only if there exists a hammock*

$$\begin{array}{ccccccc} & & X & \xrightarrow{f} & T & \xleftarrow{w} & Y \\ & \swarrow \text{Id} & \downarrow & & \downarrow & & \searrow \text{Id} \\ X & \xleftarrow{\quad} & \tilde{X} & \xrightarrow{h} & U & \xleftarrow{\quad} & \tilde{Y} \longrightarrow Y \\ & \nwarrow \text{Id} & \uparrow & & \uparrow & & \nearrow \text{Id} \\ & & X & \xrightarrow{g} & S & \xleftarrow{u} & Y \end{array}, \quad (3.7)$$

*relating  $F$  to  $G$  and where all maps except  $f, g$  and  $h$  are equivalences.*

### 3.3 Additive descent categories

**DEFINITION 3.3.1.** An additive simplicial descent category is a simplicial descent category that is also an additive category and such that the simple functor is additive.

A functor of additive simplicial descent categories is a functor of simplicial descent categories which is also additive.

**PROPOSITION 3.3.2.** *If  $\mathcal{D}$  is an additive simplicial descent category then  $\text{Ho}\mathcal{D}$  is an additive category, and the localization functor  $\gamma : \mathcal{D} \rightarrow \text{Ho}\mathcal{D}$  is additive. In addition, every functor  $F : \mathcal{D} \rightarrow \mathcal{D}'$  of additive simplicial descent categories gives rise to an additive functor  $\text{Ho}F : \text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{D}'$ .*

Throughout this section, we will assume that  $\mathcal{D}$  is an additive simplicial descent category.

**LEMMA 3.3.3.** *If  $\hat{f}, \hat{g} : X \rightarrow Y$  are morphisms in  $Ho\mathcal{D}$  then there exists zig-zags representing  $\hat{f}$  and  $\hat{g}$  with “common denominator”*

$$\begin{aligned}\rho_{\hat{f}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{f} L \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y \\ \rho_{\hat{g}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{g} L \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y.\end{aligned}$$

*Proof.* Let  $\hat{f}$  and  $\hat{g}$  be the respective classes of the following zig-zags  $F$  and  $G$

$$X \xleftarrow{\lambda_X} RX \xrightarrow{f'} T \xleftarrow{u} RY \xrightarrow{\lambda_Y} Y \quad ; \quad X \xleftarrow{\lambda_X} RX \xrightarrow{g'} S \xleftarrow{v} RY \xrightarrow{\lambda_Y} Y .$$

If  $L = cyl(u, v)$ , then  $I_T : RT \rightarrow L$  and  $I_S : RS \rightarrow L$  are equivalences by 2.4.1. Let  $w : RY \rightarrow L$  be the equivalence defined as the composition

$$RY \xrightarrow{\mu_Y} R^2Y \xrightarrow{Rv} RS \xrightarrow{I_S} L .$$

On one hand, by 3.1.11 we have that  $F$  is related to the zig-zag  $F^v$  given by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rf'} RT \xrightarrow{I_T} cyl(u, v) \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y .$$

On the other hand, it is clear that the zig-zag  $RG$  (see 3.1.9) is related to

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{Rg'} RS \xrightarrow{I_S} L \xleftarrow{I_S} RS \xleftarrow{Rv} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y .$$

Indeed, we have the hammock

$$\begin{array}{ccccccc} & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rg'} & RS & \xleftarrow{Rv} & R^2Y & \xleftarrow{\mu_Y} & RY & & \\ & Id \swarrow & & \downarrow Id & & & \downarrow Id & & & & \downarrow Id & Id \searrow & \\ RX & & & & & & & & & & & & RY \\ & Id \swarrow & & \downarrow Id & & & \downarrow Id & & & & \downarrow Id & Id \searrow & \\ & & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{Rg'} & RS & \xrightarrow{I_S} & L & \xleftarrow{w} & RY & & \end{array}$$

Hence, the proof is finished.  $\square$

**DEFINITION 3.3.4 (Definition of sum in  $Ho\mathcal{D}$ ).**

Let  $\hat{f}, \hat{g} : X \rightarrow Y$  be morphisms in  $Ho\mathcal{D}$  and  $\rho_{\hat{f}}, \rho_{\hat{g}}$  be zig-zags representing them as in 3.3.3. We define  $\hat{f} + \hat{g} : X \rightarrow Y$  as the class of

$$\rho_{\hat{f}} + \rho_{\hat{g}} : \quad X \xleftarrow{\lambda_X} RX \xrightarrow{f+g} L \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y .$$

**LEMMA 3.3.5.** *The sum of  $f$  and  $g$  in  $Ho\mathcal{D}$  is well defined. Equivalently,  $\hat{f} + \hat{g}$  does not depend on the zig-zags  $\rho_{\hat{f}}$  and  $\rho_{\hat{g}}$  with “common denominator” chosen for  $f$  and  $g$ .*

*Proof.* Consider two different zig-zags  $\rho_{\hat{f}}, \rho_{\hat{f}'}$  representing  $\hat{f}$ , as well as  $\rho_{\hat{g}}, \rho_{\hat{g}'}$  representing  $\hat{g}$

$$\begin{aligned}\rho_{\hat{f}} : X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f} S \xleftarrow{u} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \quad ; \quad \rho'_{\hat{f}} : X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f'} T \xleftarrow{v} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \\ \rho_{\hat{g}} : X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{g} S \xleftarrow{u} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \quad ; \quad \rho'_{\hat{g}} : X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{g'} T \xleftarrow{v} \mathbf{R}Y \xrightarrow{\lambda_Y} Y.\end{aligned}$$

We must check that the following zig-zags are related

$$\begin{aligned}\rho_{\hat{f}} + \rho_{\hat{g}} : X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f+g} S \xleftarrow{u} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \\ \rho'_{\hat{f}} + \rho'_{\hat{g}} : X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{f'+g'} T \xleftarrow{v} \mathbf{R}Y \xrightarrow{\lambda_Y} Y.\end{aligned}$$

We will see that both are related to a zig-zag  $\rho'_{\hat{f}} + \rho_{\hat{g}}$ .

The equivalences  $\mathbf{R}u : \mathbf{R}^2Y \rightarrow \mathbf{R}S$  and  $\mathbf{R}v : \mathbf{R}^2Y \rightarrow \mathbf{R}T$  induce equivalences

$$I_{\mathbf{R}S} : \mathbf{R}^2S \rightarrow \text{cyl}(\mathbf{R}v, \mathbf{R}u), \quad I_{\mathbf{R}T} : \mathbf{R}^2T \rightarrow \text{cyl}(\mathbf{R}v, \mathbf{R}u).$$

Consider the morphisms  $\tilde{f}' = I_{\mathbf{R}T} \circ \mathbf{R}^2f'$ ,  $\tilde{g} = I_{\mathbf{R}S} \circ \mathbf{R}^2g : \mathbf{R}^3X \rightarrow \text{cyl}(\mathbf{R}v, \mathbf{R}u)$ .

Then  $\rho'_{\hat{f}} + \rho_{\hat{g}}$  is defined by the zig-zag

$$X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\hat{\mu}_X} \mathbf{R}^3X \xrightarrow{\tilde{f}'+\tilde{g}} \text{cyl}(\mathbf{R}v, \mathbf{R}u) \xleftarrow{I_{\mathbf{R}S}} \mathbf{R}^2S \xleftarrow{\mathbf{R}^2u} \mathbf{R}^3Y \xleftarrow{\hat{\mu}_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y$$

where  $\hat{\mu}_X = \mu_X \circ \mathbf{R}\mu_X$ , and  $\hat{\mu}_Y = \mu_Y \circ \mathbf{R}\mu_Y$ .

Consider the hammocks relating  $\rho_{\hat{f}}$  to  $\rho_{\hat{f}'}$ , and  $\rho_{\hat{g}}$  to  $\rho_{\hat{g}'}$

$$\begin{array}{ccccc} & \mathbf{R}^2X & \xrightarrow{\mathbf{R}f} & \mathbf{R}S & \xleftarrow{\mathbf{R}u} \mathbf{R}^2Y \\ & \swarrow Id & \downarrow p & \downarrow p' & \downarrow p'' & \searrow Id \\ \mathbf{R}^2X & \xleftarrow{\alpha} \overline{X} & \xrightarrow{h} & M & \xleftarrow{w} \overline{Y} & \xrightarrow{\beta} \mathbf{R}^2Y \\ & \swarrow Id & \downarrow q & \downarrow q' & \downarrow q'' & \searrow Id \\ & \mathbf{R}^2X & \xrightarrow{\mathbf{R}f'} & \mathbf{R}T & \xleftarrow{\mathbf{R}v} \mathbf{R}^2Y \end{array} \quad \begin{array}{ccccc} & \mathbf{R}^2X & \xrightarrow{\mathbf{R}g} & \mathbf{R}S & \xleftarrow{\mathbf{R}u} \mathbf{R}^2Y \\ & \swarrow Id & \downarrow s & \downarrow s' & \downarrow s'' & \searrow Id \\ \mathbf{R}^2X & \xleftarrow{\alpha'} \tilde{X} & \xrightarrow{h'} & N & \xleftarrow{w'} \tilde{Y} & \xrightarrow{\beta'} \mathbf{R}^2Y \\ & \swarrow Id & \downarrow t & \downarrow t' & \downarrow t'' & \searrow Id \\ & \mathbf{R}^2X & \xrightarrow{\mathbf{R}g'} & \mathbf{R}T & \xleftarrow{\mathbf{R}v} \mathbf{R}^2Y \end{array}.$$

We will denote them by  $\mathcal{H}$  and  $\mathcal{H}'$  respectively.

*First step:*  $\rho'_{\hat{f}} + \rho_{\hat{g}}$  is related to  $\rho_{\hat{f}} + \rho_{\hat{g}}$ .

The hammocks  $\mathcal{H}$  and  $\mathcal{H}'$  give rise to the commutative diagram

$$\begin{array}{ccccccc} \mathbf{R}^2X & \xrightarrow{\mathbf{R}f} & \mathbf{R}S & \xleftarrow{\mathbf{R}u} \mathbf{R}^2Y & \xrightarrow{\mathbf{R}u} & \mathbf{R}S & \xleftarrow{\mathbf{R}g} \mathbf{R}^2X \\ p \downarrow & & p' \downarrow & & p'' \downarrow & Id \downarrow & p \downarrow \\ \overline{X} & \xrightarrow{h} & M & \xleftarrow{w} \overline{Y} & \xrightarrow{u'} & \mathbf{R}S & \xleftarrow{l} \overline{X} \\ q \uparrow & & q' \uparrow & & q'' \uparrow & Id \uparrow & q \uparrow \\ \mathbf{R}^2X & \xrightarrow{\mathbf{R}f'} & \mathbf{R}T & \xleftarrow{\mathbf{R}v} \mathbf{R}^2Y & \xrightarrow{\mathbf{R}u} & \mathbf{R}S & \xleftarrow{\mathbf{R}g} \mathbf{R}^2X \end{array}$$

where  $u'$  is the composition  $\bar{Y} \xrightarrow{\beta} R^2Y \xrightarrow{Ru} RS$  and  $l$  is  $\bar{X} \xrightarrow{\alpha} R^2X \xrightarrow{Rg} RS$ . Then, applying the functor  $cyl$  to the two rows in the middle of the above diagram we get

$$\begin{array}{ccccccc}
R^3X & \xrightarrow{R^2f} & R^2S & \xrightarrow{J_{RS}} & cyl(Ru, Ru) & \xleftarrow{K_{RS}} & R^2S \xleftarrow{R^2g} R^3X \\
Rp \downarrow & & Rp' \downarrow & & r \downarrow & & Id \downarrow \quad Rp \downarrow \\
R\bar{X} & \xrightarrow{Rh} & RM & \xrightarrow{I_M} & cyl(u', w) & \xleftarrow{L_{RS}} & R^2S \xleftarrow{Rl} R\bar{X} \\
Rq \uparrow & & Rq' \uparrow & & r' \uparrow & & Id \uparrow \quad Rq \uparrow \\
R^3X & \xrightarrow{R^2f'} & R^2T & \xrightarrow{I_{RT}} & cyl(Rv, Ru) & \xleftarrow{I_{RS}} & R^2S \xleftarrow{R^2g} R^3X
\end{array}$$

that becomes, after composing arrows, in

$$\begin{array}{ccccc}
R^3X & \xrightarrow{\bar{f}} & cyl(Ru, Ru) & \xleftarrow{\bar{g}} & R^3X \\
Rp \downarrow & & r \downarrow & & Rp \downarrow \\
R\bar{X} & \xrightarrow{\bar{h}} & cyl(u', w) & \xleftarrow{\bar{l}} & R\bar{X} \\
Rq \uparrow & & r' \uparrow & & Rq \uparrow \\
R^3X & \xrightarrow{\tilde{f}'} & cyl(Rv, Ru) & \xleftarrow{\tilde{g}} & R^3X.
\end{array}$$

Then, it holds that  $r \circ (\bar{f} + \bar{g}) = (\bar{h} + \bar{l}) \circ Rp$  and  $r' \circ (\tilde{f}' + \tilde{g}) = (\bar{h} + \bar{l}) \circ Rq$ . Therefore, the following diagram commutes

$$\begin{array}{ccccccc}
R^3X & \xrightarrow{\bar{f} + \bar{g}} & cyl(Ru, Ru) & \xleftarrow{K_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y \\
Rp \downarrow & & r \downarrow & & Id \downarrow & & Id \downarrow \\
R\bar{X} & \xrightarrow{\bar{h} + \bar{l}} & cyl(u', w) & \xleftarrow{L_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y \\
Rq \uparrow & & r' \uparrow & & Id \uparrow & & Id \uparrow \\
R^3X & \xrightarrow{\tilde{f}' + \tilde{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y.
\end{array} \tag{3.8}$$

As usual, we complete this diagram to a hammock through the natural transformations  $\lambda$  and  $\mu$ . Indeed, consider the diagrams

$$\begin{array}{ccc}
RX & \xrightarrow{\mu_X} & R^2X \xrightarrow{R\mu_X} R^3X \\
\swarrow Id & \tau \downarrow & \eta \downarrow \quad Rp \downarrow \\
RX & \xleftarrow{\varrho} R\bar{X} \xrightarrow{Id} R\bar{X} \xrightarrow{Id} R\bar{X} \\
\swarrow Id & \tau' \uparrow & \eta' \uparrow \quad Rq \uparrow \\
RX & \xrightarrow{\mu_X} & R^2X \xrightarrow{R\mu_X} R^3X
\end{array}
\quad
\begin{array}{ccc}
R^3Y & \xleftarrow{R\mu_Y} & R^2Y \xleftarrow{\mu_Y} RY \\
Id \downarrow & Id \downarrow & Id \downarrow \quad Id \searrow \\
R^3Y & \xleftarrow{R\mu_Y} & R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{Id} RY \\
Id \uparrow & Id \uparrow & Id \uparrow \quad Id \nearrow \\
R^3Y & \xleftarrow{R\mu_Y} & R^2Y \xleftarrow{\mu_Y} RY
\end{array} \tag{3.9}$$

where  $\eta = Rp \circ R\mu_X$ ,  $\eta' = Rq \circ R\mu_X$ ,  $\tau = Rp \circ R\mu_X \circ \mu_X$ ,  $\tau' = Rq \circ R\mu_X \circ \mu_X$  and  $\varrho = \lambda_{RX} \circ R\lambda_{RX} \circ R\alpha$ .

Then, the squares in the left diagram commute by definition of the arrows involved in them. In addition,  $\varrho \circ \tau = \varrho \circ \tau' = Id_{RX}$ , since

$$\varrho \circ \tau = \lambda_{RX} \circ R \lambda_{RX} \circ (R\alpha \circ Rp) \circ R\mu_X \circ \mu_X = \lambda_{RX} \circ (R\lambda_{RX} \circ R\mu_X) \circ \mu_X = \lambda_{RX} \circ \mu_X = Id_{RX}.$$

The equality  $\varrho \circ \tau' = Id_{RX}$  is checked analogously. Therefore, the diagrams in (3.9) are commutative. Attaching them to (3.8) we obtain the hammock

$$\begin{array}{ccccccccccc} & & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\bar{f}+\bar{g}} & cyl(Ru, Ru) & \xleftarrow{K_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \\ & Id \swarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & Id \searrow \\ RX & \xleftarrow{\lambda_X} & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\bar{f}+\bar{g}} & cyl(u', w) & \xleftarrow{K_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \\ & Id \swarrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & Id \searrow \\ & & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\bar{f}'+\bar{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \end{array}$$

relating  $\rho'_{\hat{f}} + \rho_{\hat{g}}$  to the zig-zag  $\tilde{\rho}$

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\hat{\mu}_X} R^3X \xrightarrow{\bar{f}+\bar{g}} cyl(Ru, Ru) \xleftarrow{K_{RS}} R^2S \xleftarrow{R^2u} R^3Y \xleftarrow{\hat{\mu}_Y} RY \xrightarrow{\lambda_Y} Y.$$

On the other hand, applying twice 3.1.9, it follows that  $\rho_{\hat{f}} + \rho_{\hat{g}}$  is related to the zig-zag  $R^2(\rho_{\hat{f}} + \rho_{\hat{g}})$ , given by

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\hat{\mu}_X} R^3X \xrightarrow{R^2(f+g)} R^2S \xleftarrow{R^2u} R^3Y \xleftarrow{\hat{\mu}_Y} RY \xrightarrow{\lambda_Y} Y$$

and  $R^2(f+g) = R^2f + R^2g$  since  $R$  is additive. In addition, by the properties of the cylinder functor we have an equivalence  $\theta : cyl(Ru, Ru) \rightarrow R^2S$  such that  $\theta \circ K_{RS} = \theta \circ J_{RS} = Id_{R^2S}$ .

Hence  $\theta \circ (\bar{f} + \bar{g}) = \theta \circ (J_{RS} \circ R^2f + K_{RS} \circ R^2g) = R^2f + R^2g$ . Therefore, we get the following hammock relating  $\tilde{\rho}$  to  $R^2(\rho_{\hat{f}} + \rho_{\hat{g}})$

$$\begin{array}{ccccccccccc} & & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\bar{f}+\bar{g}} & cyl(Ru, Ru) & \xleftarrow{K_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \\ & Id \swarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & Id \searrow \\ R^2X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{R^2(f+g)} & R^2S & \xleftarrow{Id} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \\ & Id \swarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & Id \searrow \\ & & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{R^2(f+g)} & R^2S & \xleftarrow{Id} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \end{array}$$

that finishes the proof of the first step.

*Second step:*  $\rho'_{\hat{f}} + \rho'_{\hat{g}}$  is related to  $\rho'_{\hat{f}} + \rho_{\hat{g}}$ .

Consider this time the following commutative diagram induced by the ham-

mocks  $\mathcal{H}$  and  $\mathcal{H}'$

$$\begin{array}{ccccccc}
R^2X & \xrightarrow{Rg} & RS & \xleftarrow{Ru} & R^2Y & \xrightarrow{Rv} & RT \xleftarrow{Rf'} R^2X \\
s \downarrow & & s' \downarrow & & s'' \downarrow & & Id \downarrow \\
\tilde{X} & \xrightarrow{h'} & N & \xleftarrow{w'} & \tilde{Y} & \xrightarrow{v'} & RT \xleftarrow{l'} \tilde{X} \\
t \uparrow & & t' \uparrow & & t'' \uparrow & & Id \uparrow \\
R^2X & \xrightarrow{Rg'} & RT & \xleftarrow{Rv} & R^2Y & \xrightarrow{Rv} & RT \xleftarrow{Rf'} R^2X
\end{array}$$

where  $v'$  is the composition  $\tilde{Y} \xrightarrow{\beta'} R^2Y \xrightarrow{Rv} RT$  and  $l'$  is  $\tilde{X} \xrightarrow{\alpha'} R^2X \xrightarrow{Rf'} RT$ .  
Again, applying *cyl* (this time changing the order of the arrows) we obtain

$$\begin{array}{ccccccc}
R^3X & \xrightarrow{R^2g} & R^2S & \xrightarrow{I_{RS}} & cyl(Rv, Ru) & \xleftarrow{I_{RT}} & R^2T \xleftarrow{R^2f'} R^3X \\
Rs \downarrow & & Rs' \downarrow & & k \downarrow & & Id \downarrow \\
R\tilde{X} & \xrightarrow{Rh'} & RN & \xrightarrow{I_N} & cyl(v', w') & \xleftarrow{L_{RT}} & R^2T \xleftarrow{Rl'} R\tilde{X} \\
Rt \uparrow & & Rt' \uparrow & & k' \uparrow & & Id \uparrow \\
R^3X & \xrightarrow{R^2g'} & R^2T & \xrightarrow{K_{RT}} & cyl(Rv, Rv) & \xleftarrow{J_{RT}} & R^2T \xleftarrow{R^2f'} R^3X
\end{array}$$

that becomes, after composing arrows, in

$$\begin{array}{ccc}
R^3X & \xrightarrow{\tilde{g}} & cyl(Rv, Ru) \xleftarrow{\tilde{f}'} R^3X \\
Rs \downarrow & & k \downarrow \\
R\tilde{X} & \xrightarrow{\tilde{h}} & cyl(v', w') \xleftarrow{\tilde{l}} R\tilde{X} \\
Rt \uparrow & & k' \uparrow \\
R^3X & \xrightarrow{\tilde{g}'} & cyl(Rv, Rv) \xleftarrow{\tilde{f}'} R^3X,
\end{array}$$

and again we deduce the commutative diagram

$$\begin{array}{ccccccc}
R^3X & \xrightarrow{\tilde{f}'+\tilde{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y \\
Rs \downarrow & & k \downarrow & & Id \downarrow & & Id \downarrow \\
R\tilde{X} & \xrightarrow{\tilde{h}+\tilde{l}} & cyl(v', w') & \xleftarrow{L_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y \\
Rt \uparrow & & k' \uparrow & & Id \uparrow & & Id \uparrow \\
R^3X & \xrightarrow{\tilde{f}'+\tilde{g}'} & cyl(Rv, Rv) & \xleftarrow{J_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y.
\end{array}$$

As in the previous step, it is possible to complete this diagram to the hammock

$$\begin{array}{ccccccccccc}
& & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\tilde{f}'+\tilde{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY \\
& Id \swarrow & \downarrow & & Rs \downarrow & & k \downarrow & & Id \downarrow & & Id \downarrow & & \downarrow \\
RX & \leftarrow & R\tilde{X} & \xrightarrow{\tilde{h}+\tilde{l}} & cyl(v', w') & \xleftarrow{L_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY & \rightarrow & RY \\
& Id \swarrow & \uparrow & & Rt \uparrow & & k' \uparrow & & Id \uparrow & & Id \uparrow & & \uparrow \\
& & RX & \xrightarrow{\hat{\mu}_X} & R^3X & \xrightarrow{\tilde{f}'+\tilde{g}'} & cyl(Rv, Rv) & \xleftarrow{J_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y & \xleftarrow{\hat{\mu}_Y} & RY
\end{array}$$

relating the zig-zags  $\widehat{\rho}$  and  $\overline{\rho}$  consisting respectively of the top and bottom rows of this hammock.

Note that  $\widehat{\rho} \sim \rho'_{\widehat{f}} + \rho_{\widehat{g}}$ . Indeed, we have the hammock

$$\begin{array}{ccccccccccccccc}
 & & RX & \xrightarrow{\widehat{\mu}_X} & R^3X & \xrightarrow{\widehat{f}' + \widehat{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RT}} & R^2T & \xleftarrow{R^2v} & R^3Y & \xleftarrow{\widehat{\mu}_Y} & RY & & \\
 & \swarrow Id & \downarrow & & \downarrow Id & & \downarrow Id & & & & \downarrow I_{R^2Y} & & \downarrow & \searrow Id & \\
 RX & \xleftarrow{Id} & R\widetilde{X} & \xrightarrow{\widehat{f}' + \widehat{g}} & R^3X & \xrightarrow{\widehat{f}' + \widehat{g}} & cyl(Rv, Ru) & \xleftarrow{H} & cyl(R^2Y) & \xleftarrow{Id} & cyl(R^2Y) & \xrightarrow{Id} & RY & & \\
 & \swarrow Id & \uparrow & & \uparrow Id & & \uparrow Id & & & & \uparrow J_{R^2Y} & & \uparrow & \searrow Id & \\
 & & RX & \xrightarrow{\widehat{\mu}_X} & R^3X & \xrightarrow{\widehat{f}' + \widehat{g}} & cyl(Rv, Ru) & \xleftarrow{I_{RS}} & R^2S & \xleftarrow{R^2u} & R^3Y & \xleftarrow{\widehat{\mu}_Y} & RY & & 
 \end{array}$$

where  $H$  is the morphism provided by 2.4.9. If  $\varphi : cyl(R^2Y) \rightarrow R^3Y$  is such that  $\varphi \circ I_{R^2Y} = \varphi \circ J_{R^2Y} = Id_{R^3Y}$  (see 2.2.9), and  $\widehat{\lambda}_X = R\lambda_{RX} \circ \lambda_{RX}$ , then the triangle in the right hand side of the previous hammock consists of

$$\begin{array}{c}
 RX \\
 \swarrow Id \quad \uparrow \widehat{\lambda}_X \quad \searrow Id \\
 RY \xrightarrow{\widehat{\mu}_X} R^3X \xrightarrow{I_{R^2Y}} cyl(R^2Y) \xleftarrow{J_{R^2Y}} R^3X \xleftarrow{\widehat{\mu}_X} RX \\
 \quad \quad \quad \uparrow \varphi
 \end{array}$$

Then  $\widehat{\rho} \sim \rho'_{\widehat{f}} + \rho_{\widehat{g}}$ , and the fact  $\overline{\rho} \sim \rho'_{\widehat{f}} + \rho'_{\widehat{g}}$  can be proved as in the previous step, so we are done.  $\square$

**Proof of 3.3.2.** Let us prove that the axioms of additive category are hold in  $Ho\mathcal{D}$ . We follow here the presentation given for additive categories in [GM].

*Axiom (A1):* The sum in  $Ho\mathcal{D}$  clearly makes each  $\text{Hom}_{Ho\mathcal{D}}(X, Y)$  into an abelian group.

Let us check that the composition  $\text{Hom}_{Ho\mathcal{D}}(Z, X) \times \text{Hom}_{Ho\mathcal{D}}(X, Y) \rightarrow \text{Hom}_{Ho\mathcal{D}}(Z, Y)$  is bilinear, that is,  $\widehat{h} \circ (\widehat{f} + \widehat{g}) = \widehat{h} \circ \widehat{f} + \widehat{h} \circ \widehat{g}$ ;  $(\widehat{f} + \widehat{g}) \circ \widehat{h} = \widehat{f} \circ \widehat{h} + \widehat{g} \circ \widehat{h}$ .

We will see that  $(\widehat{f} + \widehat{g}) \circ \widehat{h} = \widehat{f} \circ \widehat{h} + \widehat{g} \circ \widehat{h}$  (the other equality can be proved similarly).

Consider the morphisms  $\widehat{f}, \widehat{g} : X \rightarrow Y$  and  $\widehat{h} : Z \rightarrow X$  in  $Ho\mathcal{D}$ . As we saw before, we can assume that these morphisms are represented respectively by

$$\begin{aligned}
 \rho_{\widehat{f}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{f} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y \\
 \rho_{\widehat{g}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{g} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y \\
 \rho_{\widehat{h}} : Z &\xleftarrow{\lambda_Z} RZ \xrightarrow{h} L \xleftarrow{t} RX \xrightarrow{\lambda_X} X.
 \end{aligned}$$

The functor  $cyl$  provides the diagrams

$$\begin{array}{ccc} R^2X & \xrightarrow{Rf} & RT \\ Rt \downarrow & & \downarrow I_T \\ RL & \xrightarrow{I_L} & C \end{array} \quad ; \quad \begin{array}{ccc} R^2X & \xrightarrow{Rg} & RT \\ Rt \downarrow & & \downarrow J_T \\ RL & \xrightarrow{J_L} & C' \end{array} ,$$

where the arrows  $i_T, j_T \in E$ . Again, we have the commutative diagram in  $HoD$

$$\begin{array}{ccc} R^2T & \xrightarrow{RI_T} & RC \\ RJ_T \downarrow & & \downarrow I_C \\ RC' & \xrightarrow{I_{C'}} & N \end{array} ,$$

where all maps are in  $E$ .

Let  $u : R^3Y \rightarrow N$  be the composition  $R^3Y \xrightarrow{R^2w} R^2T \xrightarrow{RI_T} RC \xrightarrow{I_C} N$ . Set  $f' = I_C \circ RI_L$  and  $g' = I_{C'} \circ RJ_L$ . Assume that the following zig-zags represent  $\hat{f}$  and  $\hat{g}$  respectively

$$\begin{aligned} \rho'_{\hat{f}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{R\mu_X} R^3X \xrightarrow{f' \circ R^2t} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y \\ \rho'_{\hat{g}} : X &\xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{R\mu_X} R^3X \xrightarrow{g' \circ R^2t} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y . \end{aligned}$$

In this case,  $\hat{f} + \hat{g}$  is given by the zig-zag

$$X \xleftarrow{\lambda_X} RX \xrightarrow{\mu_X} R^2X \xrightarrow{R\mu_X} R^3X \xrightarrow{R^2t} R^2L \xrightarrow{f'+g'} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y$$

and if we compose it with the zig-zag  $R^2\rho_{\hat{h}}$  representing  $\hat{h}$

$$Z \xleftarrow{\lambda_Z} RZ \xrightarrow{\mu_Z} R^2Z \xrightarrow{R\mu_Z} R^3Z \xrightarrow{R^2h} R^2L \xleftarrow{R^2t} R^3X \xleftarrow{R\mu_X} R^2X \xleftarrow{\mu_X} RX \xrightarrow{\lambda_X} X$$

then by 3.1.12 we can delete arrows getting that  $(\hat{f} + \hat{g}) \circ \hat{h}$  is given by

$$Z \xleftarrow{\lambda_Z} RZ \xrightarrow{\mu_Z} R^2Z \xrightarrow{R\mu_Z} R^3Z \xrightarrow{R^2h} R^2L \xrightarrow{f'+g'} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y$$

On the other hand, we can use again  $R^2\rho_{\hat{h}}$  to compute  $\hat{f} \circ \hat{h}$  and  $\hat{g} \circ \hat{h}$ . After deleting arrows, they are given by

$$\begin{aligned} Z &\xleftarrow{\lambda_Z} RZ \xrightarrow{\mu_Z} R^2Z \xrightarrow{R\mu_Z} R^3Z \xrightarrow{R^2h} R^2L \xrightarrow{f'} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y \\ Z &\xleftarrow{\lambda_Z} RZ \xrightarrow{\mu_Z} R^2Z \xrightarrow{R\mu_Z} R^3Z \xrightarrow{R^2h} R^2L \xrightarrow{g'} N \xleftarrow{u} R^3Y \xleftarrow{R\mu_Y} R^2Y \xleftarrow{\mu_Y} RY \xrightarrow{\lambda_Y} Y \end{aligned}$$



and therefore their sum is  $(\hat{f} + \hat{g}) \circ \hat{h}$ , so in this case (A1) holds.

It remains to see that  $\rho'_{\hat{f}}$  and  $\rho'_{\hat{g}}$  represent  $\hat{f}$  and  $\hat{g}$  respectively. Considering the zig-zags  $\rho_{\hat{f}}$  and  $\rho_{\hat{g}}$ , and applying 3.1.10 to the equivalence  $t : \mathbf{R}X \rightarrow L$ , it follows that  $\hat{f}$  and  $\hat{g}$  are given by

$$\begin{aligned} X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2X \xrightarrow{I_L \circ \mathbf{R}t} C \xleftarrow{I_T \circ \mathbf{R}w} \mathbf{R}^2Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \\ X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2X \xrightarrow{J_L \circ \mathbf{R}t} C' \xleftarrow{J_T \circ \mathbf{R}w} \mathbf{R}^2Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y. \end{aligned}$$

Set  $\tau = \mathbf{R}I_T \circ \mathbf{R}^2w$  and  $\tau' = \mathbf{R}J_T \circ \mathbf{R}^2w$ . By 3.1.9 these zig-zags are related respectively to

$$\begin{aligned} X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2X \xrightarrow{\mathbf{R}\mu_X} \mathbf{R}^3X \xrightarrow{\mathbf{R}I_L \circ \mathbf{R}^2t} \mathbf{R}C \xleftarrow{\tau} \mathbf{R}^3Y \xleftarrow{\mathbf{R}\mu_Y} \mathbf{R}^2Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y \\ X &\xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2X \xrightarrow{\mathbf{R}\mu_X} \mathbf{R}^3X \xrightarrow{\mathbf{R}J_L \circ \mathbf{R}^2t} \mathbf{R}C' \xleftarrow{\tau'} \mathbf{R}^3Y \xleftarrow{\mathbf{R}\mu_Y} \mathbf{R}^2Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y. \end{aligned}$$

In the first case, it suffices to replace  $\mathbf{R}C$  by  $\mathbf{R}C \xrightarrow{I_C} N \xleftarrow{I_C} \mathbf{R}C$ , obtaining in this way  $\rho'_{\hat{f}}$ .

In the second case, replacing  $\mathbf{R}C'$  by  $\mathbf{R}C' \xrightarrow{I_{C'}} N \xleftarrow{I_{C'}} \mathbf{R}C'$  we deduce that  $\rho'_{\hat{g}}$  is related to

$$\rho''_{\hat{g}} : X \xleftarrow{\lambda_X} \mathbf{R}X \xrightarrow{\mu_X} \mathbf{R}^2X \xrightarrow{\mathbf{R}\mu_X} \mathbf{R}^3X \xrightarrow{g' \circ \mathbf{R}^2t} N \xleftarrow{u'} \mathbf{R}^3Y \xleftarrow{\mathbf{R}\mu_Y} \mathbf{R}^2Y \xleftarrow{\mu_Y} \mathbf{R}Y \xrightarrow{\lambda_Y} Y.$$

where  $u' = I_{C'} \circ \mathbf{R}J_T \circ \mathbf{R}^2w$ . But  $I_{C'} \circ \mathbf{R}J_T$  is “homotopic” to  $I_C \circ \mathbf{R}I_T$ , that is, 2.4.9 provides  $H : \text{cyl}(\mathbf{R}T) \rightarrow N$  such that composing with the inclusions of  $\mathbf{R}^2T$  into  $\text{cyl}(\mathbf{R}T)$  we obtain just  $I_{C'} \circ \mathbf{R}J_T$  and  $I_C \circ \mathbf{R}I_T$ .

Then, a hammock relating  $\rho''_{\hat{g}}$  to  $\rho'_{\hat{g}}$  can be constructed in the usual way using  $H$ . So (A1) is already proved.

*Axiom (A2):* We must show that  $\text{Ho}\mathcal{D}$  has a zero object, that is, an object 0 such that  $\text{Hom}_{\text{Ho}\mathcal{D}}(0, 0) = \{*\} = \text{trivial group}$ . Since  $\mathcal{D}$  has a zero object  $0_{\mathcal{D}}$ , that is at the same time an initial and final object, the from 3.1.15 (or 3.1.16) (A2) follows.

*Axiom (A3):* Given objects  $X, Y$  in  $\text{Ho}\mathcal{D}$ , we must show the existence of an object  $Z$  and morphisms

$$X \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{i_1} \end{array} Z \begin{array}{c} \xleftarrow{p_2} \\ \xleftarrow{i_2} \end{array} Y$$

such that  $p_1 \circ i_1 = \text{Id}_X$ ;  $p_2 \circ i_2 = \text{Id}_Y$ ;  $i_1 \circ p_1 + i_2 \circ p_2 = \text{Id}_Z$  and  $p_2 \circ i_1 = p_1 \circ i_2 = 0$  in  $\text{Ho}\mathcal{D}$ .

But, since  $X$  and  $Y$  are objects in  $\mathcal{D}$ , the data  $Z, i_1, i_2, p_1$  and  $p_2$  so exists in  $\mathcal{D}$ , and it is enough to take the image under  $\gamma : \mathcal{D} \rightarrow Ho\mathcal{D}$  of these morphisms. Hence, since  $\gamma$  is functorial it is clear that  $\gamma(p_1) \circ \gamma(i_1) = Id_X$  and  $\gamma(p_2) \circ \gamma(i_2) = Id_Y$ .

On the other hand, it follows from the definitions of sum in  $Ho\mathcal{D}$  and of  $\gamma$  that  $\gamma(f) + \gamma(g) = \gamma(f + g)$ , then  $\gamma(i_1) \circ \gamma(p_1) + \gamma(i_2) \circ \gamma(p_2) = Id_Z$ . To finish, the morphism  $0$  in  $\mathcal{D}$  is defined as the unique morphism that factors through the zero object  $0_{\mathcal{D}}$ , and since  $\gamma(0_{\mathcal{D}}) = 0_{Ho\mathcal{D}}$ , we deduce that  $\gamma$  maps the morphism  $0$  into the morphism  $0$ , so the equality  $\gamma(p_2) \circ \gamma(i_1) = \gamma(p_1) \circ \gamma(i_2) = 0$  holds, and (A3) is proven.

In addition, as mentioned before,  $\gamma : \text{Hom}_{\mathcal{D}}(X, Y) \rightarrow \text{Hom}_{Ho\mathcal{D}}(X, Y)$  is lineal, so  $\gamma$  is additive.

Finally, let  $F : \mathcal{D} \rightarrow \mathcal{D}'$  be a functor of additive simplicial descent categories. Assume given  $\hat{f}, \hat{g} : X \rightarrow Y$  in  $Ho\mathcal{D}$ , and zig-zags representing them as in 3.3.3. By definition, we have that  $\hat{f} + \hat{g}$  is represented by

$$\rho_{\hat{f}} + \rho_{\hat{g}} : \quad X \xleftarrow{\lambda_X} RX \xrightarrow{f+g} L \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y .$$

Then, by 3.1.14 it holds that  $\hat{f} + \hat{g} = \gamma(\lambda_Y) \circ (\gamma(w))^{-1} \circ \gamma(f + g) \circ (\gamma(\lambda_X))^{-1}$ , and since  $\gamma$  and  $F$  preserve sums, then  $HoF(\hat{f} + \hat{g}) = HoF(\hat{f}) + HoF(\hat{g})$  follows from the equality  $HoF \circ \gamma = \gamma \circ F$ .  $\square$

## Chapter 4

# Relationship with triangulated categories

The aim of this chapter is to describe the “left unstable” triangulated structure existing on the homotopy category  $Ho\mathcal{D}$  associated with any simplicial descent category  $\mathcal{D}$ .

The distinguished triangles will be those defined through the cone functor  $c : Maps(\mathcal{D}) \rightarrow \mathcal{D}$ .

With no extra assumption, neither additivity, this class of triangles will satisfy all axioms of triangulated category but TR3, that is the one involving the shift of distinguished triangles. When  $\mathcal{D}$  is an additive category then  $Ho\mathcal{D}$  is “right triangulated” or “suspended” [KV], so if in addition the shift functor is an equivalence of categories then  $Ho\mathcal{D}$  is a triangulated category.

For triangulated categories we will follow the notations of [GM].

In order to simplify the notations we will write  $f : X \rightarrow Y$  to denote the morphism  $\gamma(f)$  in  $Ho\mathcal{D}$ , for any morphism  $f$  in  $\mathcal{D}$ .

**DEFINITION 4.1.1** (shift functor).

The *shift functor*  $T : \mathcal{D} \rightarrow \mathcal{D}$  is defined as

$$T(X) = cyl(1 \leftarrow X \rightarrow 1) = c(X \rightarrow 1).$$

As usual, by  $X[n]$  we mean  $T^n(X)$ .

**REMARK 4.1.2.** It follows from 2.2.6 that  $T$  preserve equivalences, that is,  $T(E) \subseteq E$  (in fact we will see that  $T^{-1}(E) = E$ ).

Then  $T$  induces a functor between the localized categories, that we will denote also by  $T : Ho\mathcal{D} \rightarrow Ho\mathcal{D}$ . If  $f$  is a morphism in  $Ho\mathcal{D}$  we will also write  $f[1]$

instead of  $Tf$ .

Note that our shift functors  $T : \mathcal{D} \rightarrow \mathcal{D}$ ,  $T : Ho\mathcal{D} \rightarrow Ho\mathcal{D}$  may not be equivalences of categories.

**(4.1.3)** If  $X$  is an object of  $\mathcal{D}$ , there exists an isomorphism  $\theta_X : R(X[1]) \rightarrow (RX)[1]$  in  $Ho\mathcal{D}$  functorial in  $X$ , given by

$$R(X[1]) \xrightarrow{\lambda_{X[1]}} X[1] \xleftarrow{\lambda_X[1]} (RX)[1] .$$

The next proposition and its corollary will not be used in this work, we introduce them just for completeness.

**PROPOSITION 4.1.4.** *If  $f$  is a morphism in  $\mathcal{D}$ , there exists an isomorphism in  $Ho\mathcal{D}$*

$$\theta_f : c(f[1]) \rightarrow (c(f))[1] ,$$

*functorial on  $f$  and such that the following diagram commutes (in  $Ho\mathcal{D}$ )*

$$\begin{array}{ccc} R(Y[1]) & \xrightarrow{I_{Y[1]}} & c(f[1]) \\ \theta_Y \downarrow & & \theta_f \downarrow \\ (RY)[1] & \xrightarrow{I_Y[1]} & (c(f))[1] . \end{array}$$

*Proof.* Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{D}$ . To see the existence of the isomorphism  $\theta_f$  it suffices to apply the factorization property of the cone, 2.3.5, to the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ Id \downarrow & & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

and substitute  $c(Id_1)$  by 1 when needed.

Note that  $\theta_Y$  is just  $\theta_{0 \rightarrow Y}$ , and by definition  $\theta_f$  is functorial on  $f$ . Hence the equality  $\theta_f \circ I_{Y[1]} = I_Y[1] \circ \theta_Y$  holds.  $\square$

**COROLLARY 4.1.5.** *A morphism  $f$  of  $\mathcal{D}$  is in  $E$  if and only if  $f[1]$  is so. Therefore, a morphism  $f$  of  $Ho\mathcal{D}$  is an isomorphism if and only if  $f[1]$  is so.*

*Proof.* Consider a morphism  $f : X \rightarrow Y$  of  $\mathcal{D}$  such that  $f[1] \in E$ . From the acyclicity axiom (or its corollary 2.2.10) we deduce that  $c(f[1]) \simeq (c(f))[1] \rightarrow 1$  is in  $E$ . But  $(c(f))[1] = c(c(f) \rightarrow 1)$ , and again from (SDC 7) follows that

$c(f) \rightarrow 1$  is an equivalence, hence  $f \in E$ .

The last statement is a formal consequence of the first one since  $E$  is saturated.  $\square$

**DEFINITION 4.1.6** (triangles in  $Ho\mathcal{D}$ ).

A triangle in  $Ho\mathcal{D}$  is a sequence of morphisms in  $Ho\mathcal{D}$  of the form

$$X \rightarrow Y \rightarrow Z \rightarrow X[1].$$

A morphism between the triangles  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  and  $X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]$  is a commutative diagram in  $Ho\mathcal{D}$

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \alpha \downarrow & & \beta \downarrow & & \delta \downarrow & & \alpha[1] \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

(4.1.7) Given a morphism  $f : X \rightarrow Y$  in  $\mathcal{D}$ , if we apply *cyl* to the diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & \longleftarrow & & & \\ 1 & & & & \\ \downarrow & & Id \downarrow & & \downarrow \\ & \longleftarrow & X & \longrightarrow & 1 \\ 1 & & & & \end{array}$$

we will obtain, by 2.4.10, the diagram

$$\begin{array}{ccccc} & & RX & \xrightarrow{Rf} & RY \\ & \swarrow & \downarrow I_1 & \searrow I_Y & \downarrow \\ R1 & \xrightarrow{\quad} & c(f) & & \\ \downarrow & & \downarrow p & & \downarrow \\ & & RX & \xrightarrow{\quad} & R1, \\ \downarrow & \swarrow & \downarrow J_1 & \searrow & \\ R1 & \xrightarrow{K_1} & X[1] & & \end{array} \quad (4.1)$$

where all faces commute in  $\mathcal{D}$  except the top and bottom ones, that commutes in  $Ho\mathcal{D}$ .

Therefore,  $f$  gives rise to the following sequence of morphisms of  $\mathcal{D}$

$$RX \xrightarrow{Rf} RY \xrightarrow{I_Y} c(f) \xrightarrow{p_f} X[1].$$

It holds that the compositions  $P \circ I_Y$  and  $I_Y \circ f$  are trivial in  $Ho\mathcal{D}$ , that is, they factor through the final object 1 (since  $1 \simeq R1$  in  $Ho\mathcal{D}$ ).

**DEFINITION 4.1.8** (distinguished triangles in  $Ho\mathcal{D}$ ).

Define *distinguished triangles* in  $Ho\mathcal{D}$  as those triangles isomorphic, for some morphism  $f$  of  $\mathcal{D}$ , to

$$X \xrightarrow{f} Y \xrightarrow{\iota_Y} c(f) \xrightarrow{p} X[1] \quad (4.2)$$

where  $\iota_Y$  is the composition in  $Ho\mathcal{D}$  of  $Y \xrightarrow{\lambda_Y^{-1}} RY \xrightarrow{I_Y} c(f)$ .

**REMARK 4.1.9.** Therefore, we have automatically that the composition of two consecutive maps in a distinguished triangle is trivial, that is, it factors through the object 1 in  $Ho\mathcal{D}$ .

Next we will see that most of the axioms of triangulated categories hold for this class of distinguished triangles.

**PROPOSITION 4.1.10 (TR 1).**

- i) the triangle  $X \xrightarrow{Id} X \rightarrow 1 \rightarrow X[1]$  is distinguished, where the map  $1 \rightarrow X[1]$  is the composition  $1 \xrightarrow{\lambda_1^{-1}} R1 \xrightarrow{K_1} X[1]$  (see (4.1)).
- ii) Every triangle isomorphic to a distinguished triangle is also distinguished.
- iii) Given  $f : X \rightarrow Y$  in  $Ho\mathcal{D}$ , there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] .$$

*Proof.*

To see i), consider (4.1) for  $f = Id_X$ . Set  $\rho = K_1 \circ \lambda_1^{-1} : 1 \rightarrow X[1]$ . The map  $I_1 : R1 \rightarrow c(Id)$  is in E, so  $c(Id) \xrightarrow{\eta} 1$  is so (since  $\eta \circ I_1 = \lambda_1 \in E$ ). The diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & c(Id) & \xrightarrow{pId} & X[1] \\ Id \downarrow & & Id \downarrow & & \eta \downarrow & & Id \downarrow \\ X & \longrightarrow & X & \longrightarrow & 1 & \xrightarrow{\rho} & X[1]. \end{array}$$

provides a morphism of triangle. Indeed, to see the commutativity in  $Ho\mathcal{D}$  of the right square just note that by (4.1),  $K_1 = p \circ I_1$ , and then  $\rho \circ \eta = K_1 \circ (\lambda_1^{-1} \circ \eta) = K_1 \circ I_1^{-1} = p$ .

ii) holds by definition of distinguished triangle.

Then it remains to prove iii). Given a morphism  $f : X \rightarrow Y$  in  $Ho\mathcal{D}$ , from 3.1.5

we have that  $f$  is represented by a zig-zag of the form  $X \xleftarrow{\lambda_X} RX \xrightarrow{\bar{f}} T \xleftarrow{w} RY \xrightarrow{\lambda_Y} Y$ . If  $Z = c(\bar{f})$ , we consider the distinguished triangle

$$RX \xrightarrow{\bar{f}} T \xrightarrow{\iota_T} Z \xrightarrow{p} (RX)[1] .$$

Let  $g : Y \rightarrow Z$  and  $h : Z \rightarrow X[1]$  be the compositions given respectively by

$$\begin{aligned} Y &\xrightarrow{\lambda_Y^{-1}} \mathbf{R}Y \xrightarrow{w} T \xrightarrow{\iota_T} c(\bar{f}) = Z \\ Z &\xrightarrow{p} (\mathbf{R}X)[1] \xrightarrow{\lambda_X[1]} X[1]. \end{aligned}$$

Then, setting  $\alpha = \lambda_Y \circ w^{-1} : T \rightarrow Y$  we deduce the following commutative diagram

$$\begin{array}{ccccccc} \mathbf{R}X & \xrightarrow{\bar{f}} & T & \xrightarrow{\iota_T} & Z & \xrightarrow{p} & (\mathbf{R}X)[1] \\ \lambda_X \downarrow & & \alpha \downarrow & & Id \downarrow & & \downarrow (\lambda_X)[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

that is in fact an isomorphism of triangles, so the bottom triangle is distinguished.  $\square$

**PROPOSITION 4.1.11 (TR 3).**

If  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  and  $X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]$  are distinguished triangles and

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \downarrow \beta \\ X' & \xrightarrow{g} & Y', \end{array}$$

commutes in  $Ho\mathcal{D}$ , then there exists an isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \alpha \downarrow & & \beta \downarrow & & h \downarrow & & \alpha[1] \downarrow \\ X' & \xrightarrow{g} & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

In addition, if  $\alpha$  and  $\beta$  are isomorphisms of  $Ho\mathcal{D}$  then so is  $h$ .

*Proof.*

By definition of distinguished triangle we can assume that  $f$  and  $g$  are morphisms of  $\mathcal{D}$  and the triangles  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ ,  $X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]$  are those obtained from  $f$  and  $g$  respectively as in (4.2).

*Case 1:*  $\alpha$  and  $\beta$  are morphisms in  $\mathcal{D}$  and  $\beta \circ f = g \circ \alpha$  in  $\mathcal{D}$ .

In this case it follows directly from the functoriality of the cone the existence of  $h : c(f) = Z \rightarrow c(g) = Z'$  in  $\mathcal{D}$  such that the required diagram is commutative. If  $\alpha, \beta$  are isomorphisms in  $Ho\mathcal{D}$  then they are in  $E$ , since  $E$  is saturated. Hence, we deduce from corollary 2.2.7 that  $h \in E$ .

Case 2:  $\alpha$  and  $\beta$  are morphisms of  $\mathcal{D}$  and  $\beta f = g\alpha$  in  $Ho\mathcal{D}$ .

In this case the zig-zags

$$X \xleftarrow{\lambda_X} R^2X \xrightarrow{R(\beta \circ f)} R^2Y' \xleftarrow{Id} R^2Y' \xrightarrow{\lambda_{Y'}} Y'$$

$$X \xleftarrow{\lambda_X} R^2X \xrightarrow{R(g \circ \alpha)} R^2Y' \xleftarrow{Id} R^2Y' \xrightarrow{\lambda_{Y'}} Y'$$

define the same morphism of  $Ho\mathcal{D}$ , and by 3.1.5 we have a hammock in  $\mathcal{D}$  in the form

$$\begin{array}{ccccccc}
 & & R^2X & \xrightarrow{R^2(\beta \circ f)} & R^2Y' & \xleftarrow{Id} & R^2Y' \\
 & \swarrow Id & \downarrow w & & \downarrow t & & \downarrow \\
 R^2X & \xleftarrow{\quad} & \bar{X} & \xrightarrow{H} & L & \xleftarrow{\quad} & \bar{Y} \\
 & \nwarrow Id & \uparrow w' & & \uparrow t' & & \uparrow \\
 & & R^2X & \xrightarrow{R^2(g \circ \alpha)} & R^2Y' & \xleftarrow{Id} & R^2Y'
 \end{array} \tag{4.3}$$

where all maps are in  $\mathcal{E}$  except  $R^2(\beta \circ f)$ ,  $R^2(g \circ \alpha)$  and  $H$ . Hence, if we denote by  $\lambda^2 : R^2 \rightarrow R$  the natural transformation with  $\lambda_S^2 = \lambda_S \circ \lambda_{RS}$ , we have the following diagram consisting of the commutative squares in  $\mathcal{D}$

$$\begin{array}{ccccccccccc}
 X & \xleftarrow{\lambda_X^2} & R^2X & \xrightarrow{Id} & R^2X & \xrightarrow{w} & T & \xleftarrow{w'} & R^2X & \xrightarrow{R^2\alpha} & R^2X' & \xrightarrow{\lambda_{X'}^2} & X' \\
 \downarrow f & & \downarrow R^2f & & \downarrow R^2(\beta \circ f) & & \downarrow H & & \downarrow R^2(g \circ \alpha) & & \downarrow R^2g & & \downarrow g \\
 Y & \xleftarrow{\lambda_Y^2} & R^2Y & \xrightarrow{R^2\beta} & R^2Y' & \xrightarrow{t} & S & \xleftarrow{t'} & R^2Y' & \xrightarrow{Id} & R^2Y' & \xrightarrow{\lambda_{Y'}^2} & Y'
 \end{array}$$

By the first case there exists morphisms  $\tilde{\lambda}$ ,  $\tilde{\beta}$ ,  $u$ ,  $u'$ ,  $\tilde{\alpha}$  and  $\hat{\lambda}$  such that the



diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z = c(f) & \longrightarrow & X[1] \\
\lambda_X^2 \uparrow & & \lambda_Y^2 \uparrow & & \tilde{\lambda} \uparrow & & \lambda_X^2[1] \uparrow \\
R^2 X & \xrightarrow{R^2 f} & R^2 Y & \longrightarrow & c(R^2 f) & \longrightarrow & (R^2 X)[1] \\
Id \downarrow & & R^2 \beta \downarrow & & \tilde{\beta} \downarrow & & Id \downarrow \\
R^2 X & \xrightarrow{R^2(\beta \circ f)} & R^2 Y' & \longrightarrow & c(R^2(\beta f)) & \longrightarrow & (R^2 X)[1] \\
w \downarrow & & t \downarrow & & u \downarrow & & w[1] \downarrow \\
T & \xrightarrow{H} & S & \longrightarrow & c(H) & \longrightarrow & T[1] \\
w' \uparrow & & t' \uparrow & & u' \uparrow & & w'[1] \uparrow \\
R^2 X & \xrightarrow{R^2(g \circ \alpha)} & R^2 Y' & \longrightarrow & c(R^2(g \alpha)) & \longrightarrow & (R^2 X)[1] \\
R^2 \alpha \downarrow & & Id \downarrow & & \tilde{\alpha} \downarrow & & (R^2 \alpha)[1] \downarrow \\
R^2 X' & \xrightarrow{R^2 g} & R^2 Y' & \longrightarrow & c(R^2 g) & \longrightarrow & (R^2 X')[1] \\
\lambda_{X'}^2 \downarrow & & \lambda_{Y'}^2 \downarrow & & \hat{\lambda} \downarrow & & \lambda_{X'}^2[1] \downarrow \\
X' & \xrightarrow{g} & Y' & \longrightarrow & Z' = c(g) & \longrightarrow & X'[1]
\end{array}$$

commutes in  $Ho\mathcal{D}$ . On the other hand, the morphisms  $\tilde{\lambda}$ ,  $u$ ,  $u'$  and  $\hat{\lambda}$  are in  $E$ . Observe that the composition in  $Ho\mathcal{D}$  of the morphisms in the first column is just  $\alpha$ .

Indeed, from (4.3) it follows that  $w' = w^{-1}$ , and it is enough to have into account the equality  $\lambda_{X'}^2 \circ R^2 \alpha = \alpha \circ \lambda_X^2$ , that holds since  $\lambda^2$  is a natural transformation. In the same way, the second column is the morphism  $\beta$ , whereas the fourth one is  $\alpha[1]$ .

Summing all up, we get a morphism  $h = \hat{\lambda} \circ \tilde{\alpha} \circ (u')^{-1} \circ u \circ \tilde{\beta} \circ \tilde{\lambda}^{-1}$  such that the requested diagram commutes.

Finally, if  $\alpha$  and  $\beta$  are in  $E$ , then  $R^2 \alpha$  and  $R^2 \beta$  are also equivalences, and by the previous case the same holds for  $\tilde{\alpha}$  and  $\tilde{\beta}$ . Therefore  $h$  an isomorphism in  $Ho\mathcal{D}$ .

*Case 3:* General case:  $\alpha$  and  $\beta$  are morphism in  $Ho\mathcal{D}$ .

Let  $A$  and  $B$  be zig-zags representing  $\alpha$  and  $\beta$  respectively, given by

$$\begin{array}{ccccccc}
X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\alpha'} & S & \xleftarrow{u} & RX' \xrightarrow{\lambda_{X'}} X' \\
Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\beta'} & T & \xleftarrow{v} & RY' \xrightarrow{\lambda_{Y'}} Y' .
\end{array}$$

Consider the diagram

$$\begin{array}{ccccccc}
X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\alpha'} & S & \xleftarrow{u} & RX' & \xrightarrow{\lambda_{X'}} & X' \\
f \downarrow & & Rf \downarrow & & & & Rg \downarrow & & g \downarrow \\
Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\beta'} & T & \xleftarrow{v} & RY' & \xrightarrow{\lambda_{Y'}} & Y' .
\end{array}$$

If there exists  $t : S \rightarrow T$  such that  $t \circ \alpha' = \beta' \circ Rf$  and  $t \circ u = v \circ Rg$  in  $Ho\mathcal{D}$ , then it suffices to apply the case 2 to the squares in the above diagram. Let us check that we can always choose zig-zags representing  $\alpha$  and  $\beta$  satisfying this property. That is to say, it is enough see that there exists zig-zags  $A'$  and  $B'$

$$\begin{array}{ccccccc}
X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\alpha''} & S' & \xleftarrow{u'} & RX' & \xrightarrow{\lambda_{X'}} & X' \\
Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\beta''} & T' & \xleftarrow{v'} & RY' & \xrightarrow{\lambda_{Y'}} & Y'
\end{array}$$

representing  $\alpha$  and  $\beta$  and such that there exists  $s : S \rightarrow T$  that makes the following diagram commute in  $Ho\mathcal{D}$

$$\begin{array}{ccccccc}
X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\alpha''} & S' & \xleftarrow{u'} & RX' & \xrightarrow{\lambda_{X'}} & X' \\
f \downarrow & & Rf \downarrow & & s \downarrow & & Rg \downarrow & & g \downarrow \\
Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\beta''} & T' & \xleftarrow{v'} & RY' & \xrightarrow{\lambda_{Y'}} & Y' .
\end{array}$$

By 3.1.9, the zig-zags  $R^2A$  and  $R^2B$  given by

$$\begin{array}{ccccccccccc}
X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{R\alpha'} & RS & \xleftarrow{Ru} & R^2X' & \xleftarrow{\mu_{X'}} & RX' & \xrightarrow{\lambda_{X'}} & X' \\
Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\mu_Y} & R^2Y & \xrightarrow{R\beta'} & RT & \xleftarrow{Rv} & R^2Y' & \xleftarrow{\mu_{Y'}} & RY' & \xrightarrow{\lambda_{Y'}} & Y'
\end{array}$$

represent also the morphisms  $\alpha$  and  $\beta$  respectively.

imply that the square

$$\begin{array}{ccc}
RX' & \xrightarrow{Rg} & RY' \\
Ru \downarrow & & \downarrow I_{Y'} \\
RS & \xrightarrow{I_S} & cyl(g, u)
\end{array}$$

commutes in  $Ho\mathcal{D}$ , and  $I_{Y'}$  is an equivalence. Moreover, in the same way as before we can build the square

$$\begin{array}{ccc}
R^2Y' & \xrightarrow{Rv} & RT \\
RI_{Y'} \downarrow & & \downarrow I_T \\
Rcyl(g, u) & \xrightarrow{I_{cyl(g, u)}} & cyl(Rv, I_{Y'})
\end{array}$$

that commutes in  $Ho\mathcal{D}$  and such that all maps are in  $E$ . Set  $T' = cyl(Rv, I_{Y'})$ . Since  $I_T : RT \rightarrow T' \in E$ , it is clear that  $R^2A$  is related to

$$Y \xleftarrow{\lambda_Y} RY \xrightarrow{\mu_Y} R^2Y \xrightarrow{R\beta'} RT \xrightarrow{I_T} T' \xleftarrow{I_T} RT \xleftarrow{Rv} R^2Y' \xleftarrow{\mu_{Y'}} RY' \xrightarrow{\lambda_{Y'}} Y'.$$

Consequently it suffices to check that the morphism  $s = I_{cyl(g,u)} \circ RI_S : RT \rightarrow S'$  is such that

$$\begin{array}{ccccccccccccccc} X & \xleftarrow{\lambda_X} & RX & \xrightarrow{\mu_X} & R^2X & \xrightarrow{R\alpha'} & RS & \xleftarrow{Ru} & R^2X' & \xleftarrow{\mu_{X'}} & RX' & \xrightarrow{\lambda_{X'}} & X' \\ f \downarrow & & Rf \downarrow & & R^2f \downarrow & & s \downarrow & & R^2g \downarrow & & Rf \downarrow & & g \downarrow \\ Y & \xleftarrow{\lambda_Y} & RY & \xrightarrow{\mu_Y} & R^2Y & \xrightarrow{R\beta'} & RT & \xrightarrow{I_T} & T' & \xleftarrow{I_T} & RT & \xleftarrow{Rv} & R^2Y' & \xleftarrow{\mu_{Y'}} & RY' & \xrightarrow{\lambda_{Y'}} & Y' \end{array}$$

commutes in  $Ho\mathcal{D}$ . In order to see that, it is clear that the square II commutes. To see I, since  $\beta \circ f = g \circ \alpha$  and  $\lambda_W$  is an isomorphism in  $Ho\mathcal{D}$  for any  $W$  in  $\mathcal{D}$ , we deduce the commutativity in  $Ho\mathcal{D}$  of the diagram

$$\begin{array}{ccccc} RX & \xrightarrow{\alpha'} & S & \xrightarrow{u^{-1}} & RX' \\ Rf \downarrow & & & & Rg \downarrow \\ RY & \xrightarrow{\beta'} & T & \xrightarrow{v^{-1}} & RY' \end{array}.$$

Then we have that

$$\begin{aligned} I_T \circ R\beta' \circ R^2f &= I_T \circ Rv \circ (Rv^{-1}R\beta' \circ R^2f) = (I_T \circ Rv) \circ R^2g \circ Ru^{-1} \circ R\alpha' = \\ I_{cyl(g,u)} \circ (RI_{Y'} \circ R^2g) \circ Ru^{-1} \circ R\alpha' &= I_{cyl(g,u)} \circ RI_S \circ R^2u \circ Ru^{-1} \circ R\alpha' = s \circ R\alpha'. \end{aligned}$$

□

Now we will begin the proof of the octahedron axiom.

**(4.1.12)** Two composable morphisms  $X \xrightarrow{u} Y \xrightarrow{v} Z$  in  $\mathcal{D}$  gives rise in a natural way to the triangle

$$c(u) \xrightarrow{\alpha} c(v \circ u) \xrightarrow{\beta} c(v) \xrightarrow{\gamma} c(u)[1].$$

Indeed, applying the cone functor to the following squares

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ Id \downarrow & A & \downarrow v \\ X & \xrightarrow{v \circ u} & Z \end{array} \quad ; \quad \begin{array}{ccc} X & \xrightarrow{v \circ u} & Z \\ u \downarrow & B & \downarrow Id \\ Y & \xrightarrow{v} & Z. \end{array}$$

we obtain  $c(u) \xrightarrow{\alpha} c(vu)$  and  $c(vu) \xrightarrow{\beta} c(v)$  respectively.

On the other hand  $c(v) \xrightarrow{\gamma} c(u)[1]$  is defined as the composition  $c(v) \xrightarrow{p} Y[1] \xrightarrow{\iota_Y[1]} c(u)[1]$ .

**PROPOSITION 4.1.13.** *Under the notations given above, the triangle  $c(u) \xrightarrow{\alpha} c(v \circ u) \xrightarrow{\beta} c(v) \xrightarrow{\gamma} c(u)[1]$  is distinguished in  $Ho\mathcal{D}$ .*

*Proof.* Let us see that the above triangle is isomorphic to the one induced by  $\alpha$ , that is

$$c(u) \xrightarrow{\alpha} c(v \circ u) \xrightarrow{\iota} c(\alpha) \xrightarrow{p'} c(u)[1] .$$

We will apply the factorization property of the cone 2.3.5 to the square

$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ u \downarrow & & \downarrow v \circ u \\ Y & \xrightarrow{v} & Z . \end{array}$$

To that end we introduce some notations.

Let  $\widehat{u} : c(Id_X) \rightarrow c(v)$  be the morphism obtained by applying the cone functor by rows to the previous square, as well as  $\psi' : c(R(v \circ u)) \rightarrow c(\widehat{u})$ ,  $\psi : c(Rv) \rightarrow c(\alpha)$ ,  $\widehat{\lambda} : c(R(v \circ u)) \rightarrow c(v \circ u)$  and  $\widetilde{\lambda} : c(Rv) \rightarrow c(v)$  the morphisms obtained in the same way from the squares

$$\begin{array}{ccc} RX & \xrightarrow{R(v \circ u)} & RZ \\ I \downarrow & & \downarrow I \\ c(Id_X) & \xrightarrow{\widehat{u}} & c(v) \end{array} ; \quad \begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ I \downarrow & & \downarrow I \\ c(u) & \xrightarrow{\alpha} & c(v \circ u) \end{array} ; \quad \begin{array}{ccc} RX & \xrightarrow{R(v \circ u)} & RZ \\ \lambda_X \downarrow & & \downarrow \lambda_Z \\ X & \xrightarrow{v \circ u} & Z \end{array} ; \quad \begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ \lambda_Y \downarrow & & \downarrow \lambda_Z \\ Y & \xrightarrow{v} & Z , \end{array}$$

where each  $I$  denotes the corresponding canonical inclusion.

Denote by  $\widetilde{T} \in \Delta^\circ \mathcal{D}$  the image under  $\widetilde{Cyl}$  of  $1 \times \Delta \leftarrow C(u) \xrightarrow{\widetilde{\alpha}} C(v \circ u)$ . Take isomorphisms  $\Phi : \mathbf{s}(\widetilde{T}) \rightarrow c(\alpha)$  and  $\Psi : \mathbf{s}(\widetilde{T}) \rightarrow c(\widehat{u})$  such that the diagram

$$\begin{array}{ccccc} & R^2Z & \xrightarrow{\lambda_{RZ}} & RZ & \xleftarrow{\lambda_{RZ}} & R^2Z & \\ & \swarrow RI & \downarrow I & \swarrow I & \downarrow I & \swarrow RI & \\ Rc(v) & \xrightarrow{\lambda} & c(v) & \xleftarrow{\widetilde{\lambda}} & c(Rv) & & \\ \downarrow I & & \downarrow I & & \downarrow I & & \\ & c(R(v \circ u)) & \xrightarrow{\widehat{\lambda}} & c(v \circ u) & \xleftarrow{\lambda} & Rc(v \circ u) , & \\ \downarrow I & \swarrow \psi' & \downarrow \eta & \swarrow \eta' & \downarrow \psi & \swarrow I & \\ c(\widehat{u}) & \xleftarrow{\Psi} & \mathbf{s}\widetilde{T} & \xrightarrow{\Phi} & c(\alpha) & & \end{array} \quad (4.4)$$

commutes, where  $\eta$  is the image under  $\mathbf{s}$  of the canonical inclusion of  $C(v)$  into  $\widetilde{T}$ , whereas  $\eta'$  the image under  $\mathbf{s}$  of the morphism induced by the canonical

inclusions of  $X$  and  $Z$  into  $C(Id_X)$  and  $C(v)$  respectively.

Since  $c(Id_X) \rightarrow 1$  is an equivalence, by 2.4.1 we have that  $I : Rc(v) \rightarrow c(\widehat{u})$  is in  $\mathbf{E}$ . Hence, we deduce from the commutativity of the front face of the above diagram that  $\eta, \psi \in \mathbf{E}$ .

Set  $\tau = \psi \circ (\widetilde{\lambda})^{-1} = \Phi \circ \eta : c(v) \rightarrow c(\alpha)$ . It is enough to see that the diagram

$$\begin{array}{ccccccc} c(u) & \xrightarrow{\alpha} & c(v \circ u) & \xrightarrow{\beta} & c(v) & \xrightarrow{\gamma} & c(u)[1] \\ Id \downarrow & & Id \downarrow & (1) & \tau \downarrow & (2) & Id \downarrow \\ c(u) & \xrightarrow{\alpha} & c(v \circ u) & \xrightarrow{\iota} & c(\alpha) & \xrightarrow{p'} & c(u)[1] \end{array}$$

is a morphism of triangles. In other words, we must prove that (1) and (2) commute in  $Ho\mathcal{D}$ .

Let us see first the commutativity of (2), that is

$$\begin{array}{ccc} c(v) & \xrightarrow{p} & Y[1] \xleftarrow{\lambda_Y[1]} (RY)[1] \xrightarrow{I_Y[1]} c(u)[1] \\ \widetilde{\lambda} \uparrow & & \downarrow Id \\ c(Rv) & & \\ \psi \downarrow & & \\ c(\alpha) & \xrightarrow{p'} & c(u)[1] . \end{array}$$

Let  $p'' : c(Rv) \rightarrow (RY)[1]$  be the morphism induced by  $Rv$  (see 4.2). Then  $\lambda_Y[1] \circ p'' = p \circ \widetilde{\lambda}$  in  $\mathcal{D}$ , since both morphisms agree with the result of applying the cone functor to the following compositions

$$\begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ \lambda_Y \downarrow & & \lambda_Z \downarrow \\ Y & \xrightarrow{v} & Z \\ Id \downarrow & & \downarrow \\ Y & \longrightarrow & 1 \end{array} \quad \begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ Id \downarrow & & \downarrow \\ RY & \longrightarrow & 1 \\ \lambda_Y \downarrow & & \downarrow \\ Y & \longrightarrow & 1 . \end{array}$$

Hence, it remains to see that  $p' \circ \psi = I_Y[1] \circ p''$ , but again this equality holds in  $\mathcal{D}$  because both morphisms are equal to the image under the cone functor of the compositions

$$\begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ I_Y \downarrow & & I_Z \downarrow \\ c(u) & \xrightarrow{\alpha} & c(v \circ u) \\ Id \downarrow & & \downarrow \\ c(u) & \longrightarrow & 1 \end{array} \quad \begin{array}{ccc} RY & \xrightarrow{Rv} & RZ \\ Id \downarrow & & I_Z \downarrow \\ RY & \longrightarrow & 1 \\ I_Y \downarrow & & \downarrow \\ c(u) & \longrightarrow & 1 . \end{array}$$

Now we study the square (1), that consist of

$$\begin{array}{ccc}
c(v \circ u) & \xrightarrow{\beta} & c(v) \\
\downarrow Id & & \downarrow \eta \\
& & \mathbf{s}\tilde{T} \\
& & \downarrow \Phi \\
c(v \circ u) & \xleftarrow{\lambda_{c(v \circ u)}} Rc(v \circ u) \xrightarrow{I_{c(v \circ u)}} & c(\alpha) .
\end{array}$$

The strategy will be the following. We will define a simplicial morphism  $\Theta : \tilde{T} \rightarrow C(v)$  such that

a) If  $i_{C(v \circ u)} : C(v \circ u) \rightarrow \tilde{T}$  is the canonical inclusion and  $\tilde{\beta} : C(v \circ u) \rightarrow C(v)$  the simplicial morphism defined through the diagram B of (4.1.12), then  $\Theta \circ i_{C(v \circ u)} = \tilde{\beta}$ .

b) If  $\rho : C(v) \rightarrow \tilde{T}$  is the map induced by the canonical inclusions of  $Y$  and  $Z$  into  $C(u)$  and  $C(v \circ u)$ , then  $\Theta \circ \rho = Id_{C(v)}$ .

Assume that a) and b) are satisfied. Since  $\mathbf{s}(\rho) = \eta : c(v) \rightarrow \mathbf{s}\tilde{T}$  is an equivalence,  $\theta = \mathbf{s}\Theta = (\eta)^{-1}$  in  $Ho\mathcal{D}$ . On the other hand,  $\mathbf{s}i_{C(v \circ u)} = \eta'$  and  $\mathbf{s}(\tilde{\beta}) = \beta : c(v \circ u) \rightarrow c(v)$ . Hence we deduce from a) that  $\theta \circ \eta' = \beta$  in  $\mathcal{D}$ .

Therefore, (1) commutes, because on one hand  $\eta \circ \beta = \theta^{-1} \circ \beta = \eta'$ , and on the other hand, by (4.4)  $\eta' = \Phi^{-1} \circ I_{C(v \circ u)} \circ (\lambda_{c(v \circ u)})^{-1}$ .

Hence, it remains to prove a) and b). Define  $\Theta : \tilde{T} \rightarrow C(v)$  as follows.

Recall that  $C(u)$  is defined in degree  $n$  as  $Y \sqcup \coprod^n X \sqcup 1$ . Following the notations in 1.5.4 it can be described as

$$C(u)_n = Y^{u_1} \sqcup \coprod_{\sigma \in \Lambda_n} X^\sigma \sqcup 1^{u_0} .$$

Similarly, by 1.7.4,  $\tilde{T}_n = C(v \circ u)_n^{u_1} \sqcup \coprod_{\sigma \in \Lambda_n} C(u)_n^\sigma \sqcup 1^{u_0}$ , that is

$$\tilde{T}_n = (Z^{u_1, u_1} \sqcup \coprod_{\rho \in \Lambda_n} X^{\rho, u_1} \sqcup 1^{u_0, u_1}) \sqcup \coprod_{\sigma \in \Lambda_n} (Y^{u_1, \sigma} \sqcup \coprod_{\rho \in \Lambda_n} X^{\rho, \sigma} \sqcup 1^{u_0, \sigma}) \sqcup 1^{u_0, u_0}$$

where the superscripts are mute, and are just used as labels for indexing the coproduct. Define

$$\Theta_n : (Z^{u_1, u_1} \sqcup \coprod_{\rho \in \Lambda_n} X^{\rho, u_1} \sqcup 1^{u_0, u_1}) \sqcup \coprod_{\sigma \in \Lambda_n} (Y^{u_1, \sigma} \sqcup \coprod_{\rho \in \Lambda_n} X^{\rho, \sigma} \sqcup 1^{u_0, \sigma}) \sqcup 1^{u_0, u_0} \longrightarrow Z^{u_1} \sqcup \coprod_{\sigma \in \Lambda_n} Y^\sigma \sqcup 1^{u_0}$$

as the morphism whose restriction to the component  $\rho, \sigma$  is

$$\Theta_n|_{\rho, \sigma} = \begin{cases} Id : Z^{u_1, u_1} \rightarrow Z^{u_0} & \text{if } \rho = \sigma = u_1 \\ Id : Y^{u_1, \sigma} \rightarrow Y^\sigma & \text{if } \sigma \in \Lambda_n, \rho = u_1 \\ Id : 1^{u_0, \sigma} \rightarrow 1^{u_0} & \text{if } \rho = u_0 \\ u : X^{\rho, \sigma} \rightarrow Y^\sigma & \text{if } \rho, \sigma \in \Lambda_n, \sigma^{-1}(1) \subseteq \rho^{-1}(1) \\ u : X^{\rho, \sigma} \rightarrow Y^\rho & \text{if } \rho \in \Lambda_n, \sigma \neq u_0, \rho^{-1}(1) \subseteq \sigma^{-1}(1) . \end{cases}$$

Provided that  $\Theta$  is an isomorphism of simplicial objects, it is clear that a) and b) hold.

Therefore, it remains to see that  $\Theta$  is in fact a morphism between simplicial objects.

Given an order preserving map  $\nu : [m] \rightarrow [n]$ , we must check that  $\Theta_{m \circ \tilde{T}}(\nu) = [C(v)](\nu) \circ \Theta_n : \tilde{T}_n \rightarrow C(v)_m$  in  $\mathcal{D}$ .

Recall that  $[C(v)](\nu) : Z^{u_1} \sqcup \coprod_{\sigma \in \Lambda_n} Y^\sigma \sqcup 1^{u_0} \rightarrow Z^{u_1} \sqcup \coprod_{\sigma \in \Lambda_m} Y^\sigma \sqcup 1^{u_0}$  is given by (see 1.5.4)

$$[C(v)](\nu)|_\sigma = \begin{cases} Id : Y^\sigma \rightarrow Y^{\sigma\nu} & \text{if } \sigma\nu \in \Lambda_m \\ Id : Z^{u_1} \rightarrow Z^{u_1} & \text{if } \sigma = u_1 \\ v : Y^\sigma \rightarrow Z^{u_1} & \text{if } \sigma \in \Lambda_n \text{ and } \sigma\nu = u_1 \\ Id : 1^{u_0} \rightarrow 1_{u_0} & \text{if } \sigma = u_0 \\ Y^\sigma \rightarrow 1^{u_0} & \text{if } \sigma \in \Lambda_n \text{ and } \sigma\nu = u_0 . \end{cases}$$

On the other hand,  $\tilde{T}(\nu) : \tilde{T}_n \rightarrow \tilde{T}_m$  is (see 1.7.4)

$$\tilde{T}(\nu)|_\sigma = \begin{cases} [C(u)](\nu) : C(u)_n^\sigma \rightarrow C(u)_m^{\sigma\nu} & \text{if } \sigma\nu \in \Lambda \\ \tilde{\alpha}_m \circ [C(u)](\nu) : C(u)_n^\sigma \rightarrow C(v \circ u)_m^{u_1} & \text{if } \sigma \in \Lambda, \sigma\nu = u_1 \\ C(u)_n^\sigma \rightarrow 1^{u_1} & \text{if } \sigma \in \Lambda, \sigma\nu = u_0 \\ [C(v \circ u)](\theta) : C(v \circ u)_n^{u_1} \rightarrow C(v \circ u)_m^{u_1} & \text{if } \sigma = u_1 \\ Id : 1^{u_1} \rightarrow 1^{u_1} & \text{if } \sigma = u_0 . \end{cases}$$

that, by definition of the cone functor, is equal to

$$\tilde{T}(\nu)|_{\rho,\sigma} = \begin{cases} Id : X^{\rho,\sigma} \rightarrow X^{\rho\nu,\sigma\nu} & \text{if } \rho\nu \in \Lambda_n, \sigma\nu \neq u_0 \\ Id : Y^{u_1,\sigma} \rightarrow Y^{u_1,\sigma\nu} & \text{if } \sigma\nu \in \Lambda_m, \rho = u_1 \\ Id : 1^{u_0,\sigma} \rightarrow 1^{u_0,u_0} & \text{if } \rho = u_0 \\ X^{\rho,\sigma} \rightarrow 1^{u_0,\sigma\nu} & \text{if } \sigma\nu \neq u_0, \rho\nu = u_0, \rho \in \Lambda_n \\ u : X^{\rho,\sigma} \rightarrow Y^{u_1,\sigma\nu} & \text{if } \sigma\nu \in \Lambda_m, \rho\nu = u_1, \rho \in \Lambda_n \\ v : Y^{u_1,\sigma} \rightarrow Z^{u_1,u_1} & \text{if } \sigma\nu = u_1, \sigma \in \Lambda_n, \rho = u_1 \\ v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1,u_1} & \text{if } \sigma\nu = u_1, \rho \in \Lambda_n, \rho\nu = u_1 \\ Id : Z^{u_1,u_1} \rightarrow Z^{u_1,u_1} & \text{if } \sigma = \rho = u_1 \\ X^{\rho,\sigma} \rightarrow 1^{u_0,u_0} & \text{if } \sigma\nu = u_0, \sigma \neq u_0, \rho \in \Lambda_n \\ Y^{u_1,\sigma} \rightarrow 1^{u_0,u_0} & \text{if } \sigma\nu = u_0, \sigma \in \Lambda_n, \rho = u_1 . \end{cases}$$

Hence, the equality  $\Theta_m \circ \tilde{T}(\nu) = [C(v)](\nu) \circ \Theta_n$  is clearly satisfied over the components  $Z^{u_1,u_1}$ ,  $Y^{u_1,\sigma}$  and  $1^{u_0,\sigma}$  of  $\tilde{T}_n$ . Let us check it over the components of the form  $X^{\rho,\sigma}$ , with  $\rho \in \Lambda_n$  and  $\sigma \neq u_0$ .

Case  $\sigma^{-1}(1) \subseteq \rho^{-1}(1)$ .

In this case  $\sigma \neq u_1$  (otherwise  $\rho = u_1$ ), so  $\sigma \in \Lambda_n$  and we have that

$$[C(v)](\nu) \circ \Theta_n|_{X^{\rho,\sigma}} = \begin{cases} u : X^{\rho,\sigma} \rightarrow Y^{\sigma\nu} & \text{if } \sigma\nu \in \Lambda_m \\ v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1} & \text{if } \sigma\nu = u_1 \\ X^{\rho,\sigma} \rightarrow 1^{u_0} & \text{if } \sigma\nu = u_0 . \end{cases}$$

On the other hand, since  $\sigma^{-1}(1) \subseteq \rho^{-1}(1)$  then  $\nu^{-1}\sigma^{-1}(1) \subseteq \nu^{-1}\rho^{-1}(1)$ , that is,  $(\sigma\nu)^{-1}(1) \subseteq (\rho\nu)^{-1}(1)$ .

Therefore, if  $\sigma\nu \in \Lambda_m$  in particular  $(\sigma\nu)^{-1}(1) \neq \emptyset$  and consequently  $\rho\nu \neq u_0$ . If  $\rho\nu = u_1$ , by definition  $\tilde{T}(\nu)|_{X^{\rho,\sigma}} = u : X^{\rho,\sigma} \rightarrow Y^{u_1,\sigma\nu}$  and  $\Theta_m \circ \tilde{T}(\nu)|_{X^{\rho,\sigma}} = u : X^{\rho,\sigma} \rightarrow Y^{\sigma\nu}$ .

Otherwise, we have that  $\rho\nu \in \Lambda_m$  and then  $\tilde{T}(\nu)|_{X^{\rho,\sigma}} = Id : X^{\rho,\sigma} \rightarrow X^{\rho\nu,\sigma\nu}$ . As  $(\sigma\nu)^{-1}(1) \subseteq (\rho\nu)^{-1}(1)$  then  $\Theta_m \circ \tilde{T}(\nu)|_{X^{\rho,\sigma}} = u : X^{\rho,\sigma} \rightarrow Y^{\sigma\nu}$ .

Now assume that  $\sigma\nu = u_1$ . Then  $(\sigma\nu)^{-1}(1) = [m] \subseteq (\rho\nu)^{-1}(1)$  and  $\rho\nu = u_1$ , so  $\tilde{T}(\nu)|_{X^{\rho,\sigma}} = v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1,u_1}$ , and  $\Theta_m \circ \tilde{T}(\nu)|_{X^{\rho,\sigma}} = v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1}$ .

On the other hand, if  $\sigma\nu = u_0$ , it is clear that  $\Theta_m \circ \tilde{T}(\nu)|_{X^{\rho,\sigma}} : X^{\rho,\sigma} \rightarrow 1^{u_0}$ .



Case  $\rho^{-1}(1) \subseteq \sigma^{-1}(1)$ . Again by definition

$$[C(v)](\nu) \circ \Theta_n|_{X^{\rho,\sigma}} = \begin{cases} u : X^{\rho,\sigma} \rightarrow Y^{\rho\nu} & \text{if } \rho\nu \in \Lambda_m \\ v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1} & \text{if } \rho\nu = u_1 \\ X^{\rho,\sigma} \rightarrow 1^{u_0} & \text{if } \rho\nu = u_0 . \end{cases}$$

Note that  $(\rho\nu)^{-1}(1) \subseteq (\sigma\nu)^{-1}(1)$ .

If  $\rho\nu \in \Lambda_m$ , it follows that  $(\rho\nu)^{-1}(1) \neq \emptyset$ , then  $\sigma\nu \neq u_0$ .

Hence,  $\tilde{T}(\nu)|_{X^{\rho,\sigma}} = Id : X^{\rho,\sigma} \rightarrow X^{\rho\nu,\sigma\nu}$  and  $\Theta_{m \circ \tilde{T}}(\nu)|_{X^{\rho,\sigma}} = u : X^{\rho,\sigma} \rightarrow Y^{\rho\nu}$ .

If  $\rho\nu = u_1$ , we have that  $\sigma\nu = u_1$  and  $\Theta_{m \circ \tilde{T}}(\nu)|_{X^{\rho,\sigma}} = v \circ u : X^{\rho,\sigma} \rightarrow Z^{u_1}$ .

Finally, if  $\rho\nu = u_0$ , by definition  $\tilde{T}(\nu)|_{X^{\rho,\sigma}} : X^{\rho,\sigma} \rightarrow 1^{u_0,\sigma\nu}$  and  $\Theta_{m \circ \tilde{T}}(\nu)|_{X^{\rho,\sigma}} : X^{\rho,\sigma} \rightarrow 1^{u_0}$ , that finish the proof.  $\square$

In order to prove the octahedron axiom in the general case, we will need the following notations.

(4.1.14) Denote by  $f : X \xrightarrow{[1]} Y$  a morphism of the form  $f : X \rightarrow Y[1]$  of  $Ho\mathcal{D}$ . Then the distinguished triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  can be written as

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \swarrow [1] & \searrow \\ & Z & \end{array}$$

We will call “*octahedron upper half*” a diagram in  $Ho\mathcal{D}$  as in the following picture

$$\begin{array}{ccccc} X & \xrightarrow{w} & & & Z \\ & \searrow u & & \nearrow v & \\ & & Y & & \\ & \swarrow q & & \nwarrow r & \\ M & \xleftarrow{p} & & & N \end{array} \quad \begin{array}{c} [1] \uparrow t \\ [1] \downarrow s \\ [1] \end{array} \quad (4.5)$$

where the triangles labelled with the symbol  $*$  are distinguished and the two remaining commute (in  $Ho\mathcal{D}$ ).

**PROPOSITION 4.1.15 (TR 4, Octahedron axiom).**

*Every octahedron upper half can be completed to an octahedron. More precisely,*

given an octahedron upper half as 4.5, there exists a diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{w} & Z & & \\
 \uparrow & \swarrow u' & \searrow v' & & \downarrow s \\
 & [1] & * & & \\
 & \nearrow q' & \nwarrow r' & & \\
 M & \xleftarrow{p} & N & & \\
 & [1] & & & 
 \end{array}$$

where, again, the triangles labelled with  $*$  are distinguished and the others commute (in  $\text{Ho}\mathcal{D}$ ). Moreover, the following diagrams commute in  $\text{Ho}\mathcal{D}$

$$\begin{array}{ccc}
 & Z & \\
 v \nearrow & & \searrow v' \\
 Y & & Y' \\
 q \searrow & & \nearrow q' \\
 & M & 
 \end{array}
 \quad
 \begin{array}{ccc}
 & X[1] & \\
 u' \nearrow & & \searrow u[1] \\
 Y' & & Y[1] \\
 r' \searrow & & \nearrow r \\
 & N & 
 \end{array}$$

*Proof.* First, suppose that  $u$  and  $v$  are morphisms of  $\mathcal{D}$ . In this case, using the notations given in 4.2 and 4.1.12, it follows from TR3 that the given octahedron upper half is isomorphic to

$$\begin{array}{ccccc}
 X & \xrightarrow{v \circ u} & Z & & \\
 \downarrow u & & \uparrow v & & \downarrow \iota_Z \\
 & [1] & * & & \\
 & \nearrow p_u & \nwarrow p_v & & \\
 & \nearrow \iota_Y & \nwarrow \gamma & & \\
 c(u) & \xleftarrow{[1]} & c(v) & & 
 \end{array}$$

Hence, it suffices to prove that this octahedron upper half can be completed into a whole octahedron. Consider the distinguished triangle obtained from  $v \circ u$

$$X \xrightarrow{v \circ u} Z \xrightarrow{j_Z} c(v \circ u) \xrightarrow{p_{v \circ u}} X[1] .$$

Following the notations given in 4.1.12, we consider the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{v \circ u} & Z & & \\
 \downarrow p_u & & \uparrow j_Z & & \downarrow \iota_Z \\
 & [1] & * & & \\
 & \nearrow \alpha & \nwarrow \beta & & \\
 c(u) & \xleftarrow{[1]} & c(v) & & 
 \end{array}$$

I claim that the triangles labelled with  $*$  are distinguished. The upper triangle is clearly distinguished, whereas the lower one is so because of proposition 4.1.13. Since  $\alpha$  and  $\beta$  are the morphisms obtained as the image under the cone functor of the squares A and B in 4.1.12, it follows that the above triangles not labelled with  $*$  are commutative, as well as

$$\begin{array}{ccc}
 Y & \xrightarrow{v} & Z \\
 & \searrow \iota_Y & \downarrow j_Z \\
 & & c(v \circ u) \\
 & \nearrow \alpha & \uparrow \\
 & c(u) & 
 \end{array}
 \quad
 \begin{array}{ccc}
 & X[1] & \\
 p_{v \circ u} \nearrow & & \searrow u[1] \\
 c(v \circ u) & & Y[1] \\
 \beta \searrow & & \nearrow p_v \\
 & c(v) & 
 \end{array}
 ,$$

This finish the proof of TR 4 when  $u, v$  are morphisms of  $\mathcal{D}$ .

To see the general case, when  $u$  and  $v$  are morphisms of  $Ho\mathcal{D}$ , let us check that each octahedron upper half (4.5) is isomorphic to an octahedron upper half where  $u$  and  $v$  are in  $\mathcal{D}$ .

Since the triangle  $X \xrightarrow{u} Y \xrightarrow{q} M \xrightarrow{t} X[1]$  is distinguished, by definition there exists a morphism  $\bar{u} : \bar{X} \rightarrow \bar{Y}$  in  $\mathcal{D}$  and an isomorphism of triangles

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \xrightarrow{q} & M & \xrightarrow{t} & X[1] \\
 \tau \downarrow & & \tau' \downarrow & & \tau'' \downarrow & & \tau[1] \downarrow \\
 \bar{X} & \xrightarrow{\bar{u}} & \bar{Y} & \xrightarrow{\iota_{\bar{Y}}} & c(\bar{u}) & \xrightarrow{p_{\bar{u}}} & \bar{X}[1] .
 \end{array}$$

Hence, the isomorphisms  $\tau, \tau', \tau''$  provide an isomorphism between the given octahedron upper half and the following one

$$\begin{array}{ccccc}
 \bar{X} & \xrightarrow{\bar{v} \circ \bar{u}} & Z & & \\
 & \searrow \bar{u} & \nearrow \bar{v} & & \\
 [1] \uparrow p_{\bar{u}} & * & \bar{Y} & * & \downarrow s \\
 & \nearrow \iota_{\bar{Y}} & \nwarrow \bar{r} & & \\
 c(\bar{u}) & \xleftarrow{\bar{p}} & N & & 
 \end{array}$$

where  $\bar{v} = v \circ (\tau')^{-1}$ ,  $\bar{r} = \tau' \circ r$  and  $\bar{p} = \iota_{\bar{Y}} \circ \bar{r} = (\tau'')^{-1} \circ p$ .

Therefore we can assume that the morphism  $u$  in our octahedron upper half

$$\begin{array}{ccccc}
 X & \xrightarrow{w} & Z & & \\
 & \searrow u & \nearrow v & & \\
 [1] \uparrow t & * & Y & * & \downarrow s \\
 & \nearrow q & \nwarrow r & & \\
 M & \xleftarrow{p} & N & & 
 \end{array}$$

is a morphism in  $\mathcal{D}$ .

On the other hand, we deduce from theorem 3.1.5 that  $v : Y \rightarrow Z$  is represented by a zig-zag of morphisms of  $\mathcal{D}$  in the form

$$Y \xleftarrow{\lambda_Y} RY \xrightarrow{\bar{v}} T \xleftarrow{l} RZ \xrightarrow{\lambda_Z} Z, \quad l \in E.$$

Finally, let us see that the original octahedron upper half is isomorphic to

$$\begin{array}{ccccc} RX & \xrightarrow{\bar{v} \circ Ru} & T & & \\ & \searrow Ru & \nearrow \bar{v} & & \\ [1] \bar{t} \uparrow & * & RY & * & \downarrow \bar{s} \\ & \swarrow Rq & \nwarrow \hat{r} & & \\ RM & \xleftarrow[Rp]{[1]} & RN & & \end{array} \quad (4.6)$$

To this end, consider for any  $A$  in  $\mathcal{D}$  the isomorphism  $\theta_A$  of  $Ho\mathcal{D}$  defined as the composition  $R(A[1]) \xrightarrow{\lambda_{A[1]}} A[1] \xrightarrow{(\lambda_A[1])^{-1}} (RA)[1]$ .

Set  $\bar{t} = \theta_X \circ Rt : RM \rightarrow (RX)[1]$ . Then the following diagram commutes

$$\begin{array}{ccccccc} RX & \xrightarrow{Ru} & RY & \xrightarrow{Rq} & RM & \xrightarrow{\bar{t}} & (RX)[1] \\ \lambda_X \downarrow & & \lambda_Y \downarrow & & \lambda_M \downarrow & & \lambda_X[1] \downarrow \\ X & \xrightarrow{u} & Y & \xrightarrow{q} & M & \xrightarrow{t} & X[1] \end{array}$$

In the same way,  $\bar{s}$  and  $\hat{r}$  are the respective compositions

$$T \xrightarrow{l^{-1}} RZ \xrightarrow{Rs} RN$$

$$RN \xrightarrow{Rr} R(Y[1]) \xrightarrow{\theta_Y} (RY)[1]$$

that give rise to the isomorphism of triangles

$$\begin{array}{ccccccc} RY & \xrightarrow{\bar{v}} & T & \xrightarrow{Rq} & RN & \xrightarrow{\bar{t}} & (RX)[1] \\ \lambda_Y \downarrow & & \lambda_Z \circ l^{-1} \downarrow & & \lambda_N \downarrow & & \lambda_X[1] \downarrow \\ Y & \xrightarrow{w} & Z & \xrightarrow{q} & N & \xrightarrow{\hat{r}} & X[1] \end{array}$$

Therefore it is clear that (4.6) is an octahedron upper half isomorphic to the original, that finish the proof.  $\square$

In order to study the remaining axiom TR 2 of triangulated category, we need  $Ho\mathcal{D}$  to be additive, since TR 2 involves a “minus” sign. Recall that by 3.3.2, if we assume that  $\mathcal{D}$  is an additive simplicial descent category (definition 3.3.1) then so is  $Ho\mathcal{D}$ .

**PROPOSITION 4.1.16 (TR 2).**

i) Suppose that  $\mathcal{D}$  is an additive simplicial descent category.

If the triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$  is distinguished in  $\text{Ho}\mathcal{D}$ , then so is  $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ .

ii) If in addition the shift functor  $T : \text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{D}$  is fully faithful, so the converse statement also holds.

*Proof.*

Proof of i) By definition of distinguished triangle we can assume that  $u$  is a morphism of  $\mathcal{D}$ , and that  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$  is the triangle obtained from  $u$ , that is

$$X \xrightarrow{u} Y \xrightarrow{\iota_Y} c(u) \xrightarrow{p_u} X[1] .$$

We must prove that the triangle

$$Y \xrightarrow{\iota_Y} c(u) \xrightarrow{p_u} X[1] \xrightarrow{-u[1]} Y[1] .$$

is distinguished. Define the morphism  $I_Y : \text{R}Y \rightarrow c(u)$  as in (2.2.4). We will see that there exists an isomorphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{\iota_Y} & c(u) & \xrightarrow{p_u} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \uparrow \lambda_Y & & \uparrow Id & & \uparrow \theta & & \uparrow \lambda_Y[1] \\ \text{R}Y & \xrightarrow{I_Y} & c(u) & \xrightarrow{\iota_{c(u)}} & c(I_Y) & \xrightarrow{p_{I_Y}} & (\text{R}Y)[1] \end{array} \quad (4.7)$$

Let  $0$  be a zero object of  $\mathcal{D}$ , that is at the same time initial and final object. Given a simplicial object  $S$  in  $\mathcal{D}$  it holds that, at the simplicial level, the simplicial cone of  $0 \rightarrow S$  is by definition the simplicial cylinder of  $0 \leftarrow 0 \rightarrow S$ , that coincides with  $S$  (since  $0$  is the unit for the coproduct). In particular, if  $S = T \times \Delta$  for some object  $T$  of  $\mathcal{D}$ , it follows that  $c(0 \rightarrow T) = \mathbf{s}(T \times \Delta) = \text{R}T$ . By assumption  $\mathbf{s}$  is additive, so  $\text{R}0 = 0$ . Consequently, we can consider the morphism  $f : c(I_Y) \rightarrow \text{R}(X[1])$  in  $\mathcal{D}$  defined as the image under the cone functor of the square

$$\begin{array}{ccc} \text{R}Y & \xrightarrow{I_Y} & c(u) \\ \downarrow & & \downarrow p_u \\ 0 & \longrightarrow & X[1] \end{array}$$

obtained from diagram (4.1) in 4.1.7.

Let us see that  $f \in E$ . Consider the following commutative cube in  $\mathcal{D}$

$$\begin{array}{ccccc}
 & & 0 & \longrightarrow & X \\
 & \swarrow & \downarrow Id & \swarrow u & \downarrow Id \\
 Y & \xrightarrow{\quad} & Y & & Y \\
 \downarrow I & \swarrow & \downarrow & \swarrow & \downarrow \\
 & & 0 & \longrightarrow & X \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 0 & \longrightarrow & 0 & & 0
 \end{array} \tag{4.8}$$

We will apply the factorization property of the cone, 2.3.5, to the upper and lower faces of this cube. Begin with the lower one

$$\begin{array}{ccc}
 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & X.
 \end{array}$$

Applying the cone functor by rows and columns we obtain the morphisms  $RX \rightarrow 0$  and  $0 \rightarrow X[1]$  respectively.

Denote by  $X\{1\}$  the simplicial cone object associated with the morphism  $X \times \Delta \rightarrow 0 \times \Delta$ . If  $\hat{T}$  is the image under  $\widetilde{Cyl}$  of  $0 \times \Delta \leftarrow 0 \times \Delta \rightarrow X < 1 >$ , then  $\hat{T} = X\{1\}$ . Note that  $s(X\{1\}) = X[1]$  by definition of  $X[1]$ .

The natural isomorphisms  $\Psi' : X[1] \rightarrow (RX)[1]$  and  $\Phi' : X[1] \rightarrow R(X[1])$  obtained from 2.3.5 are such that the diagram

$$\begin{array}{ccccc}
 (RX)[1] & \xrightarrow{\lambda_X[1]} & X[1] & \xleftarrow{\lambda_{X[1]}} & R(X[1]), \\
 \swarrow Id & & \swarrow Id & & \swarrow Id \\
 (RX)[1] & \xleftarrow{\Psi'} & X[1] & \xrightarrow{\Phi'} & R(X[1])
 \end{array}$$

Therefore  $\Psi' = (\lambda_X[1])^{-1}$  and  $\Phi' = \lambda_{X[1]}^{-1}$ .

On the other hand, consider now

$$\begin{array}{ccc}
 0 & \longrightarrow & X \\
 \downarrow & & \downarrow u \\
 Y & \xrightarrow{Id} & Y.
 \end{array}$$

Let  $g : RX \rightarrow c(Id_Y)$  be the result of applying the cone functor by rows to the above square, and  $\tilde{T} \in \Delta^\circ \mathcal{D}$  the image under  $\widetilde{Cyl}$  of the diagram  $0 \times \Delta \leftarrow Y \times \Delta \xrightarrow{i_Y} C(u)$ .

It follows from 2.3.5 the existence of natural isomorphisms  $\Psi : s\tilde{T} \rightarrow c(g)$  and

$\Phi : \mathbf{s}\tilde{T} \rightarrow c(I_Y)$  in  $Ho\mathcal{D}$ .

Diagram (4.8) provides the morphisms

$$f : c(I_Y) \rightarrow (RX)[1] ; f' : \mathbf{s}\tilde{T} \rightarrow X[1] ; f'' : c(g) \rightarrow (RX)[1]$$

. We deduce from the functoriality of the isomorphisms  $\Psi$ ,  $\Phi$ ,  $\Psi'$  and  $\Phi'$  the commutativity in  $Ho\mathcal{D}$  of the following diagram

$$\begin{array}{ccccc} c(g) & \xleftarrow{\Phi} & \mathbf{s}\tilde{T} & \xrightarrow{\Psi} & c(I_Y) \\ f'' \downarrow & & f' \downarrow & & f \downarrow \\ (RX)[1] & \xrightarrow{\lambda_X[1]} & X[1] & \xleftarrow{\lambda_X[1]} & R(X[1]) \end{array} \quad (4.9)$$

On the other hand, the morphism  $f'' : c(g) \rightarrow (RX)[1]$  is obtained as the image under the cone functor of the square

$$\begin{array}{ccc} RX & \xrightarrow{g} & c(Id_Y) \\ Id \downarrow & & \downarrow \\ RX & \longrightarrow & 0 \end{array}$$

Since  $Id_Y \in E$ , from 2.2.10 we deduce that  $c(Id_Y) \rightarrow 0$  is an equivalence. Hence it follows from corollary 2.2.7 that  $f''$  is in  $E$ .

Then by (4.9) we find that  $f \in E$ . Take  $\theta = \lambda_{X[1]} \circ f = f'' \circ \Psi^{-1} : c(I_Y) \rightarrow X[1]$ , that is isomorphism in  $Ho\mathcal{D}$  by definition.

We must check that diagram (4.7) is in fact a morphism of triangles. In other words, we must see that the squares appearing in this diagram are commutative in  $Ho\mathcal{D}$ .

The equality  $\iota_Y \circ \lambda_Y = I_Y$ , follows from the definition of  $\iota_Y$ .

Let us prove that  $\theta \circ \iota_{c(u)} = p_u : c(u) \rightarrow X[1]$  in  $Ho\mathcal{D}$ . By definition  $f$  comes from the commutative square

$$\begin{array}{ccc} RY & \xrightarrow{I_Y} & c(u) \\ \downarrow & & \downarrow p_u \\ 0 & \longrightarrow & X[1] \end{array},$$

and hence  $f \circ I_{c(u)} = R p_u$ , so  $\theta \circ \iota_{c(u)} = \lambda_X[1] \circ f \circ I_{c(u)} \circ \lambda_{c(u)}^{-1} = \lambda_X[1] \circ R p_u \circ \lambda_{c(u)}^{-1}$ . But by the naturality of  $\lambda$ , this is just  $p_u$ .

Therefore, it remains to check the equality  $-u[1] \circ \theta = \lambda_Y[1] \circ p_{I_Y}$  in  $Ho\mathcal{D}$ . To this end, it is enough to define a simplicial morphism  $H : X\{1\} \rightarrow \tilde{T}$  such that

a)  $f' \circ \mathbf{s}H = Id_{X[1]}$ , hence  $\mathbf{s}H = (f')^{-1}$  and  $\theta^{-1} = \Psi \circ \mathbf{s}H$ .

b)  $\lambda_Y[1] \circ p_{I_Y} \circ \Psi \circ \mathbf{s}H = -u[1]$ .

By definition  $X\{1\}_n$  is the coproduct (that is, the direct sum) of  $n$  copies of  $X$ . We will index this sum over the set  $\{\sigma : [n] \rightarrow [1] \mid \sigma \neq u_0, u_1\} = \Lambda_n$  (see 1.5.4). Then

$$X\{1\}_n = \bigoplus_{\sigma \in \Lambda_n} X^\sigma .$$

On the other hand,  $\widetilde{T} = \widetilde{Cyl}(0 \times \Delta \leftarrow Y \times \Delta \xrightarrow{i_Y} C(u))$ , that in degree  $n$  is (see 1.7.4)  $\widetilde{T}_n = C(u)_n^{u_1} \sqcup \bigoplus_{\sigma \in \Lambda_n} Y^\sigma$ . Again, by definition of the simplicial cone functor,  $\widetilde{T}$  can be described as

$$\widetilde{T}_n = (Y^{u_1, u_1} \oplus \bigoplus_{\rho \in \Lambda_n} X^{\rho, u_1}) \oplus \bigoplus_{\sigma \in \Lambda_n} Y^{u_1, \sigma} .$$

Define the restriction of  $H_n : X\{1\}_n \rightarrow \widetilde{T}_n$  to the component  $\sigma$  of  $X\{1\}_n$  as the map

$$H_n|_\sigma = (Id, -u) : X^\sigma \rightarrow X^{\sigma, u_1} \oplus Y^{u_1, \sigma} .$$

Now we are ready to check a) and b).

Firstly, let us prove that  $f' \circ sH = Id_{X[1]}$ . Let  $Q : C(u) \rightarrow X\{1\}$  be the morphism obtained from the square

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ Id \downarrow & & \downarrow \\ X & \longrightarrow & 0 . \end{array}$$

By definition  $f' = sF'$ , where  $F' : \widetilde{T} \rightarrow X\{1\}$  is the morphism obtained from applying  $\widetilde{Cyl}$  to the diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & Y & \xrightarrow{i_Y} & C(u) \\ \downarrow & & \downarrow & & \downarrow Q \\ 0 & \longleftarrow & 0 & \longrightarrow & X\{1\} . \end{array}$$

Then  $F'_n : (Y^{u_1, u_1} \oplus \bigoplus_{\rho \in \Lambda_n} X^{\rho, u_1}) \oplus \bigoplus_{\sigma \in \Lambda_n} Y^{u_1, \sigma} \longrightarrow \bigoplus_{\sigma \in \Lambda_n} X^\sigma$  is given by

$$F'|_{\rho, \sigma} = \begin{cases} 0 : Y^{u_1, \sigma} \rightarrow X\{1\}_n & \text{if } \sigma \neq u_0, \rho = u_1 \\ Id : X^{\rho, u_1} \rightarrow X^\rho & \text{if } \rho \in \Lambda_n, \sigma = u_1 . \end{cases}$$

Therefore it is clear that  $F' \circ H = Id_{X\{1\}}$ , so  $f' \circ sH = Id_{X[1]}$ .

Proof of b). We may check the commutativity of

$$\begin{array}{ccccc} X[1] & \xrightarrow{-u[1]} & Y[1] & & \\ sH \downarrow & & \uparrow \lambda_Y[1] & & \\ s\widetilde{T} & \xrightarrow{\Psi} & c(I_Y) & \xrightarrow{p_{I_Y}} & (RY)[1] . \end{array}$$



Denote by  $P : \tilde{T} \rightarrow Y\{1\}$  the image under  $\widetilde{Cyl}$  of

$$\begin{array}{ccccc} 0 & \longleftarrow & Y & \xrightarrow{i_Y} & C(u) \\ \downarrow & & \downarrow Id & & \downarrow \\ 0 & \longleftarrow & Y & \longrightarrow & 0, \end{array}$$

Consider now the cube

$$\begin{array}{ccccc} & & 0 & \longrightarrow & X \\ & \swarrow & \downarrow Id & \swarrow u & \downarrow \\ Y & \xrightarrow{\quad} & Y & & \\ \downarrow Id & & \downarrow & & \downarrow \\ & \swarrow & 0 & \longrightarrow & 0 \\ & \swarrow & \downarrow & \swarrow & \\ Y & \longrightarrow & 0 & & \end{array}$$

The isomorphisms provided by 2.3.5 are natural, so the above cube gives rise as before to the following commutative diagram in  $Ho\mathcal{D}$

$$\begin{array}{ccc} s\tilde{T} & \xrightarrow{\Psi} & c(I_Y) \\ sP \downarrow & & p_{I_Y} \downarrow \\ Y[1] & \xleftarrow{\lambda_Y[1]} & (RY)[1], \end{array}$$

consequently  $\lambda_Y[1] \circ p_{I_Y} \circ \Psi = sP$  in  $Ho\mathcal{D}$ .

Moreover, we have trivially the equality of simplicial morphisms  $P \circ H = -u\{1\} : X\{1\} \rightarrow Y\{1\}$ , that is just the morphism induced by

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ -u \downarrow & & \downarrow \\ Y & \longrightarrow & 0. \end{array}$$

Then  $\lambda_Y[1] \circ p_{I_Y} \circ \Psi \circ sH = sP \circ sH = -u[1]$ , so b) is proven.

To finish the proof it remains to see that  $H$  is a morphism of simplicial objects.

Recall that  $\tilde{T}_n = (Y^{u_1, u_1} \oplus \bigoplus_{\rho \in \Lambda_n} X^{\rho, u_1}) \oplus \bigoplus_{\sigma \in \Lambda_n} Y^{u_1, \sigma}$  and if  $\alpha : [m] \rightarrow [n]$  is a morphism of  $\Delta$ , then  $\tilde{T}(\alpha) : \tilde{T}_n \rightarrow \tilde{T}_m$  is

$$\tilde{T}(\alpha)|_{\sigma} = \begin{cases} Id : Y^{\sigma} \rightarrow Y^{\sigma\alpha} & \text{if } \sigma\alpha \in \Lambda_m \\ (i_Y)_m : Y^{\sigma} \rightarrow C(u)_m^{u_1} & \text{if } \sigma \in \Lambda_n, \sigma\alpha = u_1 \\ 0 : Y^{\sigma} \rightarrow \tilde{T}_m & \text{if } \sigma \in \Lambda_n, \sigma\alpha = u_0 \\ [C(u)](\alpha) : C(u)_n^{u_1} \rightarrow C(u)_m^{u_1} & \text{if } \sigma = u_1. \end{cases}$$

That is

$$\tilde{T}(\alpha)|_{\rho,\sigma} = \begin{cases} Id : Y^{u_1,\sigma} \rightarrow Y^{u_1,\sigma\alpha} & \text{if } \sigma\alpha \neq u_0, \rho = u_1 \\ 0 : Y^{u_1,\sigma} \rightarrow \tilde{T}_m & \text{if } \sigma \in \Lambda_n, \sigma\alpha = u_0, \rho = u_1 \\ Id : X^{\rho,u_1} \rightarrow X^{\rho\alpha,u_1} & \text{if } \rho\alpha \in \Lambda_m, \sigma = u_1 \\ u : X^{\rho,u_1} \rightarrow Y^{u_1,u_1} & \text{if } \rho \in \Lambda_n \text{ and } \rho\alpha = u_1, \sigma = u_1 \\ 0 : X^{\rho,u_1} \rightarrow \tilde{T}_m & \text{if } \rho \in \Lambda_n, \rho\alpha = u_0, \alpha = u_1 . \end{cases}$$

On the other hand,  $X\{1\}_n = \bigoplus_{\sigma \in \Lambda_n} X^\sigma$ , and the restriction of  $(X\{1\})(\alpha) : X\{1\}_n \rightarrow X\{1\}_m$  to the component  $\sigma$  is defined as

$$(X\{1\})(\alpha)|_\sigma = \begin{cases} Id : X^\sigma \rightarrow X^{\sigma\alpha} & \text{if } \sigma\alpha \in \Lambda_m \\ 0 : X^\sigma \rightarrow X\{1\}_m & \text{if } \sigma\alpha \notin \Lambda_m . \end{cases}$$

In addition,  $H_n : X\{1\}_n \rightarrow \tilde{T}_n$  is  $H_n|_\sigma = (Id, -u) : X^\sigma \rightarrow X^{\sigma,u_1} \oplus Y^{u_1,\sigma}$ .

Let us see that  $H_{m \circ X\{1\}}(\alpha) = \tilde{T}(\alpha) \circ H_n$ . We have that

$$H_{m \circ X\{1\}}(\alpha)|_\sigma = \begin{cases} (Id, -u) : X^\sigma \rightarrow X^{\sigma\alpha,u_1} \oplus Y^{u_1,\sigma\alpha} & \text{if } \sigma\alpha \in \Lambda_m \\ 0 : X^\sigma \rightarrow \tilde{T}_m & \text{if } \sigma\alpha \notin \Lambda_m . \end{cases}$$

If  $\sigma\alpha \neq u_0, u_1$ , it follows from the definitions that  $\tilde{T}(\alpha) \circ H_n|_\sigma = (Id, -u) : X^\sigma \rightarrow X^{\sigma\alpha,u_1} \oplus Y^{u_1,\sigma\alpha}$ .

If  $\sigma\alpha = u_0$  then  $\tilde{T}(\alpha)|_{X^{\sigma,u_1} \oplus Y^{u_1,\sigma}}$  is the trivial morphism, so the equality is satisfied.

Finally, if  $\sigma\alpha = u_1$  then  $\tilde{T}(\alpha) \circ H_n|_\sigma$  is the composition

$$X^\sigma \xrightarrow{(Id, -u)} X^{\sigma,u_1} \oplus Y^{u_1,\sigma} \xrightarrow{u+Id_Y} Y^{u_1,u_1}$$

Then  $\tilde{T}(\alpha) \circ H_n|_\sigma = 0$ , and the proof of *i*) is finished.

Proof of *ii*). Assume that  $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$  is distinguished. Applying *ii*) twice we obtain that  $X[1] \xrightarrow{-u[1]} Y[1] \xrightarrow{-v[1]} Z[1] \xrightarrow{-w[1]} X[2]$  is also distinguished. If we take the trivial isomorphism of triangles consisting of  $\pm Id$ , we deduce that  $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v[1]} Z[1] \xrightarrow{w[1]} X[2]$  is distinguished. By the axiom TR1, the morphism  $u : X \rightarrow Y$  can be inserted into a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} X[1]$$

If we apply three times *i*) to it, we obtain the distinguished triangle  $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v'[1]} Z'[1] \xrightarrow{w'[1]} X[2]$ .

Then it follows from TR3 the existence of an isomorphism  $\Theta : Z[1] \rightarrow Z'[1]$ , such that the diagram

$$\begin{array}{ccccccc} X[1] & \xrightarrow{u[1]} & Y[1] & \xrightarrow{v[1]} & Z[1] & \xrightarrow{w[1]} & X[2] \\ Id \downarrow & & Id \downarrow & & \Theta \downarrow & & Id \downarrow \\ X[1] & \xrightarrow{u[1]} & Y[1] & \xrightarrow{v'[1]} & Z'[1] & \xrightarrow{w'[1]} & X[2] . \end{array}$$

commutes. Since  $T$  is fully faithful, there exists an isomorphism  $\Theta' : Z \rightarrow Z'$  of  $Ho\mathcal{D}$  such that  $\Theta = \Theta'[1]$ , and the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ Id \downarrow & & Id \downarrow & & \Theta' \downarrow & & Id \downarrow \\ X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X[1] . \end{array}$$

is commutative, so the upper triangle is distinguished.  $\square$

Summing all up, we have the following

**THEOREM 4.1.17.** *If  $\mathcal{D}$  is a simplicial descent category then the axioms TR1, TR3 and TR4 of triangulated category hold.*

*If moreover  $\mathcal{D}$  is an additive simplicial descent category then  $Ho\mathcal{D}$  is a “suspended” (or right triangulated, [KV]).*

*A functor  $F : \mathcal{D} \rightarrow \mathcal{D}'$  of additive simplicial descent categories 3.3.1 induces a functor of suspended categories  $F : Ho\mathcal{D} \rightarrow Ho\mathcal{D}'$ .*

*Proof.* Except the last part, the theorem is already proven.

Let  $F : \mathcal{D} \rightarrow \mathcal{D}'$  be a functor of additive simplicial descent categories, and  $\Theta : s' \circ \Delta^\circ F \rightarrow F \circ s$  a natural transformation as in definition 2.5.1.

If  $f : X \rightarrow Y$  is a morphism of  $\mathcal{D}$ , let us see that  $F(c(f))$  is isomorphic to  $c'(F(f))$  in  $Ho\mathcal{D}'$ , through a functorial isomorphism  $\theta$ .

Since  $F$  is additive and the simplicial cone is defined degreewise using direct sums, it follows that the canonical morphism  $\sigma_F : \Delta^\circ F(C(f \times \Delta)) \simeq C(F(f) \times \Delta)$  in  $\Delta^\circ \mathcal{D}'$ . On the other hand  $\sigma_F$  commutes with the canonical inclusions  $F(i_Y) : F(Y) \times \Delta \rightarrow \Delta^\circ F(C(f \times \Delta))$  and  $i_{FY} : F(Y) \times \Delta \rightarrow C(F(f) \times \Delta)$ .

Hence,  $c'(F(f)) = s'C(F(f) \times \Delta) \xrightarrow{s' \sigma_F} s' \Delta^\circ F(C(f \times \Delta)) \xrightarrow{\Theta} F(s(f \times \Delta)) = F(c(f))$  gives rise to the natural isomorphism  $\theta$  between  $c'(F(f))$  and  $F(c(f))$ . In particular  $\theta : F(X[1]) = F(c(X \rightarrow 0)) \simeq (FX)[1] = c'(FX \rightarrow 0)$  in  $Ho\mathcal{D}'$ ,

and from the functoriality of  $\Theta$  and  $\sigma_F$  follows that the diagram

$$\begin{array}{ccccccc}
FX & \xrightarrow{F(f)} & FY & \xrightarrow{F(\iota_Y)} & c'(F(f)) & \xrightarrow{F(p_f)} & F(X[1]) \\
Id \downarrow & & Id \downarrow & & \theta \downarrow & & Id \downarrow \\
FX & \xrightarrow{F(f)} & FY & \xrightarrow{F(\iota_Y)} & F(c(f)) & \xrightarrow{F(p_f)} & F(X[1]) \xrightarrow{\theta} (FX)[1] .
\end{array}$$

is commutative, providing an isomorphism of triangles.  $\square$

**COROLLARY 4.1.18.** *If  $\mathcal{D}$  is an additive simplicial descent category such that  $T : Ho\mathcal{D} \rightarrow Ho\mathcal{D}$  is an automorphism of categories, then  $Ho\mathcal{D}$  is a triangulated category.*

*A functor  $F : \mathcal{D} \rightarrow \mathcal{D}'$  of additive simplicial descent categories induces a functor of triangulated categories  $F : Ho\mathcal{D} \rightarrow Ho\mathcal{D}'$ .*

**REMARK 4.1.19.** In fact, the axioms TR1, ..., TR4 hold just assuming that  $\mathcal{D}$  is an additive simplicial descent category and  $T$  is fully faithful.

## Chapter 5

# Examples of Simplicial Descent Categories

In the previous chapters we developed the notion and properties of simplicial descent categories. Now we introduce examples of such categories. The first one consists of the category of chain complexes. The axioms of simplicial descent categories will be checked by hand in this case, whereas in the remaining examples we will use the transfer lemma to this end.

### 5.1 Chain complexes and homotopy equivalences

#### 5.1.1 Preliminaries

(5.1.1) Let  $\mathcal{A}$  be an additive category. Denote by  $Ch_*(\mathcal{A})$  the category of chain complexes in  $\mathcal{A}$ . We will assume that  $\mathcal{A}$  has numerable sums, that is, if  $\{A_k\}_{k \in \mathbb{Z}}$  is a family of objects of  $\mathcal{A}$ , then  $\bigoplus_{k \in \mathbb{Z}} A_k$  exists in  $\mathcal{A}$ .

If we consider the category of uniformly bounded below chain complexes (see 5.2.4), for instance positive chain complexes, the assumption of the existence of numerable sums in  $\mathcal{A}$  can be dropped, so in this case  $\mathcal{A}$  is just an additive category  $\mathcal{A}$ .

**DEFINITION 5.1.2** (Double and triple complexes; total functor).

- Let  $Ch_*Ch_*(\mathcal{A})$  be the category of double chain complexes, also called “naïf” (cf. [Del]). An object  $A$  of  $Ch_*Ch_*(\mathcal{A})$  consists of the data

$$A = \{A_{i_1, i_2}; d^1 : A_{i_1, i_2} \rightarrow A_{i_1-1, i_2}, d^2 : A_{i_1, i_2} \rightarrow A_{i_1, i_2-1}\}$$

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots & \leftarrow & A_{0,2} & \xleftarrow{d^1} & A_{1,2} & \xleftarrow{d^1} & A_{2,2} \leftarrow \cdots \\
& \downarrow d^2 & & \downarrow d^2 & & \downarrow d^2 & \\
\cdots & \leftarrow & A_{0,1} & \xleftarrow{d^1} & A_{1,1} & \xleftarrow{d^1} & A_{2,1} \leftarrow \cdots \\
& \downarrow d^2 & & \downarrow d^2 & & \downarrow d^2 & \\
\cdots & \leftarrow & A_{0,0} & \xleftarrow{d^1} & A_{1,0} & \xleftarrow{d^1} & A_{2,0} \leftarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots & 
\end{array}$$

The functor  $Tot : Ch_*Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  is defined as follows.

If  $A = \{A_{i_1,i_2}; d^1 : A_{i_1,i_2} \rightarrow A_{i_1-1,i_2}, d^2 : A_{i_1,i_2} \rightarrow A_{i_1,i_2-1}\}$  is a double complex,  $TotA$  is the (single) chain complex given by

$$(TotA)_n = \bigoplus_{i_1+i_2=n} A_{i_1,i_2}; \quad d = \oplus (-1)^{i_2} d^1 + d^2.$$

- Consider now the category  $Ch_*Ch_*Ch_*(\mathcal{A}) = 3-Ch_*(\mathcal{A})$  of triple chain complexes.

Given an object  $A = \{A_{i_1,i_2,i_3}; d^1, d^2, d^3\}$  of  $3-Ch_*(\mathcal{A})$ , where  $d^j$  is the boundary map corresponding to the index  $i_j$ , set

$$Tot^{1,2}(A)_{p,q} = \bigoplus_{i_1+i_2=p} A_{i_1,i_2,q}; \quad d^p = \oplus (-1)^{i_2} d^1 + d^2, \quad d^q = \oplus d^{i_3};$$

$$Tot^{2,3}(A)_{p,q} = \bigoplus_{i_2+i_3=p} A_{q,i_2,i_3}; \quad d^p = \oplus (-1)^{i_3} d^2 + d^3, \quad d^q = \oplus d^{i_1};$$

$$Tot^{1,3}(A)_{p,q} = \bigoplus_{i_1+i_3=p} A_{i_1,q,i_3}; \quad d^p = \oplus (-1)^{i_3} d^1 + d^3, \quad d^q = \oplus d^{i_2}.$$

In this way we obtain the functors  $Tot^{1,2}, Tot^{2,3}, Tot^{1,3} : 3-Ch_*(\mathcal{A}) \rightarrow Ch_*Ch_*(\mathcal{A})$ .

**REMARK 5.1.3.** Following [Del], the functor  $Tot : Ch_*Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  is the total functor corresponding to the order  $i_1 > i_2$  and the respective total functor corresponding to  $i_2 > i_1$  is canonically isomorphic to the one used here. In other words, if  $\Gamma : Ch_*Ch_*(\mathcal{A}) \rightarrow Ch_*Ch_*(\mathcal{A})$  is the functor which swaps the indexes of a double complex, then the following diagram commutes (up to

canonical isomorphism)

$$\begin{array}{ccc}
 Ch_*Ch_*(\mathcal{A}) & \xrightarrow{\Gamma} & Ch_*Ch_*(\mathcal{A}) \\
 & \searrow Tot \quad \swarrow Tot & \\
 & Ch_*(\mathcal{A}) & .
 \end{array}$$

**LEMMA 5.1.4.** *The functors  $Tot : 3 - Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  obtain by composing  $Tot : Ch_*Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  with  $Tot^{1,2}, Tot^{2,3}, Tot^{1,3}$  are canonically isomorphic.*

*Proof.* Following the notations in [Del],  $Tot \circ Tot^{1,2}$  is the total functor given by the order  $i_3 < i_2 < i_1$ , whereas  $Tot \circ Tot^{2,3}$  corresponds to  $i_1 < i_3 < i_2$  and  $Tot \circ Tot^{1,3}$  to  $i_2 < i_3 < i_1$ . So it follows from loc. cit. that these three compositions are canonically isomorphic.  $\square$

Moreover, the total functor “commutes” with cones in this case. Firstly, we will remind the classical construction of cone functor in the chain complex case, that will be also denoted by  $c : Fl(Ch_*(\mathcal{A})) \rightarrow Ch_*(\mathcal{A})$ . In fact, the functor  $c$  is obtained as a particular case of the total functor  $Tot$ .

**DEFINITION 5.1.5.** If  $\mathcal{B}$  is an additive category, the cone functor

$$c : Fl(Ch_*\mathcal{B}) \rightarrow Ch_*(\mathcal{B})$$

assigns to the morphism  $f : X \rightarrow Y$  of chain complexes the chain complex

$$c(f)_n = Y_n \oplus X_{n-1} \quad d^{cf} = \begin{pmatrix} d^Y & 0 \\ f & -d^X \end{pmatrix} .$$

Equivalently, if  $\mathcal{J} : Fl(Ch_*\mathcal{B}) \rightarrow Ch_*Ch_*\mathcal{B}$  is the functor with  $\mathcal{J}f$  equal to the double complex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longleftarrow & X_0 & \xleftarrow{d^X} & X_1 & \xleftarrow{d^X} & X_2 \longleftarrow \cdots \\
 & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \\
 \cdots & \longleftarrow & Y_0 & \xleftarrow{d^Y} & Y_1 & \xleftarrow{d^Y} & Y_2 \longleftarrow \cdots \\
 & \vdots & & \vdots & & \vdots & 
 \end{array}$$

then  $c : Fl(Ch_*\mathcal{B}) \rightarrow Ch_*(\mathcal{B})$  is the composition

$$Fl(Ch_*\mathcal{B}) \xrightarrow{\mathcal{J}} Ch_*Ch_*\mathcal{B} \xrightarrow{Tot} Ch_*\mathcal{B}.$$

**LEMMA 5.1.6.**

*i) If  $\mathcal{B} = Ch_*(\mathcal{A})$  and  $c^{\mathcal{A}} : Fl(Ch_*(\mathcal{A})) \rightarrow Ch_*(\mathcal{A})$ ,  $c^{\mathcal{B}} : Fl(Ch_*(\mathcal{B})) \rightarrow Ch_*(\mathcal{B})$  are the respective cone functors, the following diagram commutes (up to canonical isomorphism)*

$$\begin{array}{ccc} Fl(Ch_*(\mathcal{B})) & \xrightarrow{c^{\mathcal{B}}} & Ch_*(\mathcal{B}) \\ \downarrow Tot & & \downarrow Tot \\ Fl(Ch_*(\mathcal{A})) & \xrightarrow{c^{\mathcal{A}}} & Ch_*(\mathcal{A}). \end{array}$$

*ii) The functor  $Tot$  preserve homotopies. That is, if  $f, g : X \rightarrow Y$  are homotopic morphisms of chain complexes, then the induced morphism in the total complex are also homotopic.*

*Proof.*

i) It suffices to check the commutativity (up to isomorphism) of the diagrams (I) and (II) bellow

$$\begin{array}{ccccc} Fl(Ch_*Ch_*(\mathcal{A})) & \xrightarrow{\mathcal{J}^{\mathcal{B}}} & 3 - Ch_*\mathcal{A} & \xrightarrow{Tot_{1,2}} & Ch_*Ch_*(\mathcal{A}) \\ \downarrow Tot & (I) & \downarrow Tot_{1,3} & (II) & \downarrow Tot \\ Fl(Ch_*(\mathcal{A})) & \xrightarrow{\mathcal{J}^{\mathcal{A}}} & Ch_*Ch_*(\mathcal{A}) & \xrightarrow{Tot} & Ch_*(\mathcal{A}). \end{array}$$

The commutativity of (II) follows from lemma 5.1.4, whereas the commutativity of (I) is an easy computation.

Indeed, if  $f : B_{n,m} \rightarrow C_{n,m}$  is a morphism of  $Ch_*Ch_*(\mathcal{A}) = Ch_*\mathcal{B}$  then  $Tot(f) : Tot(B) \rightarrow Tot(C)$ , and  $\mathcal{J}(Tot(f))$  is the double complex  $\{A_{i_1, i_2}; d^1, d^2\}$  given by

$$\begin{cases} 0 & \text{if } i_2 > 1 \\ Tot(B)_{i_1} & \text{if } i_2 = 1 \\ Tot(C)_{i_1} & \text{if } i_2 = 0 \end{cases}$$

with boundary map  $d^1$  is equal to either  $d^{Tot(B)}$  or  $d^{Tot(C)}$  depending on the case, and  $d^2 = Tot(f)$ .

On the other hand,  $\mathcal{J}^{\mathcal{B}}(f)$  is the triple complex  $D_{i_1, i_2, i_3}$  given by

$$\begin{cases} 0 & \text{if } i_2 > 1 \\ B_{i_1, i_3} & \text{if } i_2 = 1 \\ C_{i_1, i_3} & \text{if } i_2 = 0 \end{cases}$$



The boundary map  $d^2$  is equal to  $f$ , whereas  $d^1$  is equal to the boundary map of either  $B_{n,m}$  or  $C_{n,m}$  (depending on the case) respect to the index  $n$ , and analogously  $d^3$  is the one respective to the index  $m$ . Thus

$$Tot_{1,3}(D)_{p,q} = \bigoplus_{n+m=p} D_{n,q,m} = \begin{cases} 0 & \text{if } q > 1 \\ \bigoplus_{n+m=p} B_{n,m} & \text{if } q = 1 \\ \bigoplus_{n+m=p} C_{n,m} & \text{if } q = 0 \end{cases} = \mathcal{J}(Tot(f))_{p,q}.$$

Moreover, the boundary maps coincide, since  $d^p : Tot_{1,3}(D)_{p,q} \rightarrow Tot_{1,3}(D)_{p-1,q}$  is by definition

$$d^p = \begin{cases} 0 & \text{if } q > 1 \\ \oplus(-1)^m d_B^n + d_B^m & \text{if } q = 1 \\ \oplus(-1)^m d_C^n + d_C^m & \text{if } q = 0 \end{cases} = \begin{cases} 0 & \text{if } q > 1 \\ d : Tot(B)_p \rightarrow Tot(B)_{p-1} & \text{if } q = 1 \\ d : Tot(C)_p \rightarrow Tot(C)_{p-1} & \text{if } q = 0 \end{cases}.$$

Finally,  $d^q : Tot_{1,3}(D)_{p,q} \rightarrow Tot_{1,3}(D)_{p,q-1}$  is  $\oplus d_D^{i2} = \oplus f = Tot(f)$ .

ii) can be deduced from i) having in mind that a morphism  $p : X \rightarrow Y$  of  $Ch_*(\mathcal{B})$  is homotopic to 0 if and only if  $p$  can be extended to the cone of  $X$ . Hence, if  $f$  and  $g$  are homotopic in  $Ch_*(\mathcal{B})$  then  $\exists H : c^{\mathcal{B}}(X) \rightarrow Y$  such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f-g} & Y \\ & \searrow & \nearrow H \\ & c^{\mathcal{B}}(X) & \end{array}.$$

is commutative. Applying  $Tot$  to the previous diagram, it follows from i) that  $Totf - Totg$  can be extended to  $c(TotX)$ , so the statement is proven.  $\square$

**REMARK 5.1.7.** Recall that the cone functor  $c : Fl(Ch_*(\mathcal{A})) \rightarrow Ch_*(\mathcal{A})$  satisfies the following properties

- i)  $f$  is a homotopy equivalence if and only if  $c(f)$  is contractible, that is, is and only if the morphism  $c(f) \rightarrow 0$  is a homotopy equivalence.
- ii) if  $\mathcal{A}$  is abelian,  $f$  induces an isomorphism in homology if and only if the homology of  $c(f)$  is equal to 0.

**DEFINITION 5.1.8. Simple functor:** The simple functor  $s : \Delta^\circ Ch_*(\mathcal{A}) \longrightarrow Ch_*(\mathcal{A})$  is defined as the composition

$$\Delta^\circ Ch_*(\mathcal{A}) \xrightarrow{K} Ch_* Ch_*(\mathcal{A}) \xrightarrow{Tot} Ch_*(\mathcal{A})$$

where  $K(\{X, d_i, s_j\}) = \{X, \sum(-1)^i d_i\}$ .

More explicitly, let  $X = \{X_n, d_i, s_j\}$  be a simplicial chain complex. Then, each  $X_n$  is a chain complex, that will be referred to as  $\{X_{n,p}, d_{X_n}\}_{p \in \mathbb{Z}}$ .

Note that  $X$  induces the double complex (5.1), with vertical boundary map  $d_{X_n} : X_{n,p} \rightarrow X_{n,p-1}$  and horizontal boundary map  $\partial : X_{n,p} \rightarrow X_{n-1,p}$  defined as  $\partial = \sum_{i=0}^n (-1)^i d_i$ .

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \leftarrow X_{n-1,p+1} & \xleftarrow{\partial} & X_{n,p+1} & \xleftarrow{\partial} & X_{n+1,p+1} & \leftarrow \cdots \\
 & \downarrow d_{X_{n-1}} & & \downarrow d_{X_n} & & \downarrow d_{X_{n+1}} & \\
 \cdots & \leftarrow X_{n-1,p} & \xleftarrow{\partial} & X_{n,p} & \xleftarrow{\partial} & X_{n+1,p} & \leftarrow \cdots \\
 & \downarrow d_{X_{n-1}} & & \downarrow d_{X_n} & & \downarrow d_{X_{n+1}} & \\
 \cdots & \leftarrow X_{n-1,p-1} & \xleftarrow{\partial} & X_{n,p-1} & \xleftarrow{\partial} & X_{n+1,p-1} & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & 
 \end{array} \tag{5.1}$$

Thus, the image under the simple functor of  $X$  is the chain complex  $\mathbf{s}X$  given by

$$(\mathbf{s}X)_q = \bigoplus_{p+n=q} X_{n,p} \quad d = \bigoplus (-1)^p \partial + d_{X_n} : \bigoplus_{p+n=q} X_{n,p} \longrightarrow \bigoplus_{p+n=q-1} X_{n,p}.$$

**Weak equivalences:** Define  $E$  as the class of homotopy equivalences.

**Transformation  $\lambda$ :** Given  $A \in Ch_*(\mathcal{A})$ ,  $\mathbf{s}(A \times \Delta)$  in degree  $n$  is  $\bigoplus_{k \leq n} A_k$ , in such a way that  $A$  is canonically a direct summand of  $\mathbf{s}(A \times \Delta)$ . Then,  $\lambda_A : \mathbf{s}(A \times \Delta) \rightarrow A$  is just the projection.

**Transformation  $\mu$ :** Given  $Z \in \Delta^\circ \Delta^\circ Ch_*(\mathcal{A})$ ,  $\mu_Z : \mathbf{s}D(Z) \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}(Z)$  is obtained from the Alexander-Whitney map A.1.3.

In degree  $n$ , the restriction of  $(\mu_Z)_n : \bigoplus_{p+q=n} Z_{p,p,q} \rightarrow \bigoplus_{i+j+q=n} Z_{i,j,q}$  to the component  $Z_{p,p,q}$  is  $\bigoplus_{i+j=p} \mu_{Z_{i,j,q}} : Z_{p,p,q} \rightarrow Z_{i,j,q}$ , where

$$\mu_{Z_{i,j,q}} = Z(d^0 \cdots \overset{j}{d^j} \cdots d^0, d^p d^{p-1} \cdots d^{j+1}).$$

**PROPOSITION 5.1.9.** *Let  $\mathcal{A}$  be an additive category with numerable sums. Then  $Ch_*(\mathcal{A})\mathcal{A}$  together with the homotopy equivalences, and together with the simple functor,  $\lambda$  and  $\mu$  defined above is an additive simplicial descent category. In addition,  $\mu$  is associative and  $\lambda$  is quasi-invertible.*

**Proof of 5.1.9.**

The first two axioms are well known, whereas (SDC 3) follows from the additivity of  $\mathbf{s}$  (since it is the composition of additive functors).

*Proof of axiom (SDC 4):* We must check that the diagram bellow commutes up to natural homotopy equivalence

$$\begin{array}{ccccc}
 \Delta^\circ \Delta^\circ Ch_*(\mathcal{A}) & \xrightarrow{\Delta^\circ K} & \Delta^\circ Ch_* Ch_*(\mathcal{A}) & \xrightarrow{\Delta^\circ Tot} & \Delta^\circ Ch_*(\mathcal{A}) \\
 \downarrow D & & & & \downarrow K \\
 & & & & Ch_* Ch_*(\mathcal{A}) \\
 & & & & \downarrow Tot \\
 \Delta^\circ Ch_*(\mathcal{A}) & \xrightarrow{K} & Ch_* Ch_*(\mathcal{A}) & \xrightarrow{Tot} & Ch_*(\mathcal{A}).
 \end{array}$$

Following the notations given in 5.1.2, we can split our diagram into

$$\begin{array}{ccccc}
 \Delta^\circ \Delta^\circ Ch_*(\mathcal{A}) & \xrightarrow{\Delta^\circ K} & \Delta^\circ Ch_* Ch_*(\mathcal{A}) & \xrightarrow{\Delta^\circ Tot} & \Delta^\circ Ch_*(\mathcal{A}) \\
 \downarrow D & & \downarrow K & & \downarrow K \\
 & (I) & 3 - Ch_*(\mathcal{A}) & \# & Ch_* Ch_*(\mathcal{A}) \\
 & & \downarrow Tot^{1,2} & & \downarrow Tot \\
 \Delta^\circ Ch_*(\mathcal{A}) & \xrightarrow{K} & Ch_* Ch_*(\mathcal{A}) & \xrightarrow{Tot} & Ch_*(\mathcal{A}).
 \end{array} \tag{5.2}$$

The right hand side of (5.2) is commutative, whereas (I) commutes up to homotopy, by the Eilenberg-Zilber-Cartier's theorem (see A.1.1), taking  $\mathcal{U} = Ch_*(\mathcal{A})$ . Then, given  $Z \in \Delta^\circ \Delta^\circ Ch_*(\mathcal{A})$ , consider  $\mu_{E-Z}(Z) : KD(Z) \rightarrow Tot^{1,2} K \Delta^\circ K(Z)$  as in A.1.1. Hence  $\mu = Tot \circ \mu_{E-Z} : \mathbf{s}D \rightarrow \mathbf{s} \Delta^\circ \mathbf{s}$  is a homotopy equivalence because  $Tot$  preserve homotopies.

*Proof if axiom (SDC 5):*

Firstly, consider an additive category  $\mathcal{B}$  and the functor  $\mathcal{I} : \mathcal{B} \rightarrow Ch_*(\mathcal{B})$  that maps  $A$  into the complex  $\mathcal{I}(A)_0 = A$ ,  $\mathcal{I}(A)_n = 0$  if  $n > 0$ .

Let us see that there exists a functor  $G : \mathcal{B} \rightarrow Ch_* \mathcal{B}$  such that  $G(A)$  is contractible for every  $A$ . Recall that contractible means that the identity over  $G(A)$  is homotopic to the zero morphism. In addition, we will check that  $K(- \times \Delta) = \mathcal{I} \oplus G$ .

Indeed, if  $A \in \mathcal{B}$  then  $K(A \times \Delta)$  is the chain complex

$$\cdots \longleftarrow 0 \longleftarrow A \xleftarrow{0} A \xleftarrow{Id} A \xleftarrow{0} A \xleftarrow{Id} A \longleftarrow \cdots.$$

Then, define  $G(A)$  as

$$\cdots \longleftarrow 0 \longleftarrow 0 \longleftarrow A \xleftarrow{Id} A \xleftarrow{0} A \xleftarrow{Id} A \longleftarrow \cdots.$$

Clearly,  $K(A \times \Delta) = \mathcal{I}(A) \oplus G(A)$ . To see that  $G(A)$  is contractible, just take the homotopy  $H_n = Id_A : G(A)_n \rightarrow G(A)_{n+1}$ ,  $n > 0$ .

Set now  $\mathcal{B} = Ch_*\mathcal{A}$ . If  $A$  is a chain complex of  $\mathcal{B}$ , it holds in  $Ch_*(\mathcal{B})$  that  $K(A \times \Delta) = \mathcal{I}(A) \oplus G(A)$  and then

$$\mathbf{s}(A \times \Delta) = TotK(A \times \Delta) = Tot\mathcal{I}(A) \oplus TotG(A) = A \oplus TotG(A).$$

By definition  $\lambda_A$  is the projection  $A \oplus TotG(A) \rightarrow A$ . Since  $Tot$  preserve homotopies and is additive, then  $TotG(A)$  is contractible. Hence  $\lambda_A$  is a homotopy equivalence.

*Proof of axiom (SDC 6):*

The case  $Y = 0 \times \Delta$  can be found, for instance, in [B].

The general case follows from proposition 2.1.11.

*Proof of axiom (SDC 7):*

Let  $\mathcal{B}$  an additive category. The functor  $K : \Delta^\circ \mathcal{B} \rightarrow Ch_*\mathcal{B}$  induces in a natural way a functor  $Fl(\Delta^\circ \mathcal{B}) \rightarrow Fl(Ch_*\mathcal{B})$ , that will be denoted also by  $K$ .

Assume proven the commutativity up to homotopy equivalence of the diagram bellow<sup>1</sup>

$$\begin{array}{ccc} Fl(\Delta^\circ \mathcal{B}) & \xrightarrow{C} & \Delta^\circ \mathcal{B} \\ K \downarrow & & \downarrow K \\ Fl(Ch_*\mathcal{B}) & \xrightarrow{c^{\mathcal{B}}} & Ch_*\mathcal{B}, \end{array} \quad (5.3)$$

where  $C$  denotes the simplicial cone functor 1.5.7, and  $c : Fl(Ch_*\mathcal{B}) \rightarrow Ch_*\mathcal{B}$  is the cone of chain complexes given in 5.1.5.

In this case, if  $\mathcal{B} = Ch_*(\mathcal{A})$ , since  $Tot$  preserves homotopies (see 5.1.6) we obtain that  $Tot \circ K \circ C = \mathbf{s} \circ C$  is homotopic to  $Tot \circ c^{\mathcal{B}} \circ K$ . Again by lemma 5.1.6,  $Tot \circ c^{\mathcal{B}}$  is isomorphic to  $c^{\mathcal{A}} \circ Tot$ . Therefore  $\mathbf{s} \circ C$  is homotopic to  $c^{\mathcal{A}} \circ Tot \circ K = c^{\mathcal{A}} \circ \mathbf{s}$ . Equivalently, the following diagram commutes up to (natural) homotopy equivalence

$$\begin{array}{ccc} Fl(\Delta^\circ Ch_*(\mathcal{A})) & \xrightarrow{C} & \Delta^\circ Ch_*(\mathcal{A}) \\ \mathbf{s} \downarrow & & \downarrow \mathbf{s} \\ Fl(Ch_*\mathcal{A}) & \xrightarrow{c^{\mathcal{A}}} & Ch_*\mathcal{A}. \end{array} \quad (5.4)$$

---

<sup>1</sup> This is a well known fact and due to Dold and Puppe. An analogous proof for the cosimplicial case appears in [H] p.21

Hence, given  $f : A \rightarrow B$  in  $\Delta^\circ Ch_*(\mathcal{A})$ , it follows that  $sCf \rightarrow 0$  is a homotopy equivalence if and only if  $c^A(sf) \rightarrow 0$  is so, and this happens if and only if  $sf$  is a homotopy equivalence, by the classical properties satisfied by the cone functor  $c^A : Fl(Ch_*(\mathcal{A})) \rightarrow Ch_*(\mathcal{A})$ . Consequently (SDC 7) would be proven. Hence it remains to prove the commutativity up to equivalence of diagram (5.3). Indeed, let  $f : A \rightarrow B$  be a morphism of simplicial chain complexes. By definition  $C(f)$  is the total simplicial object associated with the following biaugmented bisimplicial object (see 1.3.17)

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 B_2 \longleftarrow A_2 & \rightleftarrows A_2 & \rightleftarrows A_2 & \rightleftarrows A_2 & \rightleftarrows A_2 & \cdots & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 B_1 \longleftarrow A_1 & \rightleftarrows A_1 & \rightleftarrows A_1 & \rightleftarrows A_1 & \rightleftarrows A_1 & \cdots & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 B_0 \longleftarrow A_0 & \rightleftarrows A_0 & \rightleftarrows A_0 & \rightleftarrows A_0 & \rightleftarrows A_0 & \cdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & 0 & \rightleftarrows 0 & \rightleftarrows 0 & \rightleftarrows 0 & \cdots & 
 \end{array}$$

The image under  $K$  of  $C(f)$  is the same as the total chain complex of the double complex obtained by applying  $K$  to

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \vdots & & & \\
 A_0 & \rightleftarrows A_1 & \rightleftarrows A_2 & \rightleftarrows A_3 & \cdots & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 A_0 & \rightleftarrows A_1 & \rightleftarrows A_2 & \rightleftarrows A_3 & \cdots & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 B_0 & \rightleftarrows B_1 & \rightleftarrows B_2 & \rightleftarrows B_3 & \cdots & & 
 \end{array}$$

This double complex is homotopic by columns to

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
& A_0 & \xleftarrow{d^{KA}} & A_1 & \xleftarrow{d^{KA}} & A_2 & \longleftarrow \cdots \\
& \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \\
& B_0 & \xleftarrow{d^{KB}} & B_1 & \xleftarrow{d^{KB}} & B_2 & \longleftarrow \cdots
\end{array}$$

whose associated complex is just  $c(Kf)$ .

We will give explicitly a homotopy equivalence between  $K(C(f)) \in Ch_*\mathcal{B}$  and  $F = c(K(f))$ . We have that  $F$  is the chain complex given by

$$F_n = c(K(f))_n = B_n \oplus A_{n-1} ; \quad d^F(b, a) = (d^{KB}(b) + f(a), -d^{KA}(a)).$$

By definition, it holds that

$$K(C(f))_n = B_n \oplus A_{n-1} \oplus \cdots \oplus A_0 \quad \text{and} \quad d^{K(C(f))} \text{ is}$$

$$d^{KB_n} + f_{n-1} + \sum_{k=1}^n (-1)^k d^{KA_{n-k}} + \sum_{1 \leq k \leq n/2} Id_{A_{n-2k-1}} : K(C(f))_n \rightarrow K(C(f))_{n-1}.$$

Consider the chain complex  $\tilde{A}$  defined as

$$\begin{aligned}
\tilde{A}_n &= A_{n-2} \oplus \cdots \oplus A_0 \text{ if } n \geq 2 \text{ and } \tilde{A}_0 = \tilde{A}_1 = 0 \\
d^{\tilde{A}} &= \sum_{k=2}^n (-1)^k d^{KA_{n-k}} + \sum_{1 \leq k \leq n/2} Id_{A_{n-2k-1}} : \tilde{A}_n \rightarrow \tilde{A}_{n-1}.
\end{aligned}$$

Then  $K(C(f)) = F \oplus \tilde{A}$ , since it holds that  $d^F \oplus d^{\tilde{A}} = d^{K(C(f))}$ . In addition,  $\tilde{A}$  is contractible.

To see that, let  $h_n = \sum_{k=2}^{n+1} Id_{A_{n-k}} : A_{n-2} \oplus \cdots \oplus A_0 \rightarrow A_{n-1} \oplus A_{n-2} \oplus \cdots \oplus A_0$  if  $n \geq 2$  and  $h_0 = h_1 = 0$ . If  $n \geq 2$  then

$$h_{n-1}d^{\tilde{A}} + d^{\tilde{A}}h_n = \sum_{k=2}^n (-1)^k d^{A_{n-k}} + \sum_{1 \leq k \leq n/2} Id_{A_{n-2k-1}} + \sum_{k=3}^{n+1} (-1)^k d^{A_{n+1-k}} + \sum_{1 \leq k \leq (n+1)/2} Id_{A_{n-2k}} = Id_{\tilde{A}_n}$$

Thus, the projection  $K(C(f)) \rightarrow c(K(f))$  is a homotopy equivalence.

*Proof of axiom (SDC 8):*

The functor  $\Upsilon : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow \Delta^\circ Ch_*(\mathcal{A})$  assigns to the simplicial chain complex  $X$  the chain complex  $\Upsilon X$  whose face morphisms are  $d_i^{\Upsilon X} = d_{n-i}^X : X_n \rightarrow X_{n-1}$ . Then  $K(\Upsilon X)$  has as horizontal boundary map the morphism  $d^{K(\Upsilon X)} = \sum_{i=0}^n (-1)^i d_{n-i}^X = (-1)^n d^{KX}$ , whereas the vertical boundary maps coincide in both cases.

The double complexes  $KX$  and  $K(\Upsilon X)$  are then canonically isomorphic. Composing with  $Tot$ , we deduce that the functors  $\mathbf{s}, \mathbf{s} \circ \Upsilon : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  are also canonically isomorphic, so (SDC 8) is proven.

*Compatibility between  $\lambda$  and  $\mu$ :*

Given  $X$  in  $\Delta^\circ Ch_*(\mathcal{A})$ , we must check that the following composition of morphisms are equal to the identity

$$\begin{aligned} \mathbf{s}X &\xrightarrow{\mu_{\Delta \times X}} \mathbf{s}((\mathbf{s}X) \times \Delta) \xrightarrow{\lambda_{\mathbf{s}X}} \mathbf{s}X \\ \mathbf{s}X &\xrightarrow{\mu_{X \times \Delta}} \mathbf{s}(n \rightarrow \mathbf{s}(X_n \times \Delta)) \xrightarrow{\mathbf{s}(\lambda_{X_n})} \mathbf{s}X . \end{aligned}$$

Firstly, consider the bisimplicial chain complex  $Z = \Delta \times X$ . By definition

$$\mathbf{s}((\mathbf{s}X) \times \Delta)_n = \bigoplus_{p+q=n} \mathbf{s}(X)_q = \bigoplus_{p+i+j=n} X_{i,j}$$

and  $(\lambda_{\mathbf{s}X})_n : \bigoplus_{p+q=n} (\mathbf{s}X)_p \rightarrow (\mathbf{s}X)_n$  is the projection of  $\bigoplus_{p+i+j=n} X_{i,j}$  onto  $\bigoplus_{i+j=n} X_{i,j}$ .

On the other hand, the restriction of  $(\mu_Z)_n : \bigoplus_{l+k=n} X_{l,k} \rightarrow \bigoplus_{s+t+k=n} X_{t,k}$  to  $X_{l,k}$  is

$$\bigoplus_{s+t=l} X(d^l d^{l-1} \dots d^{t+1}) : X_{l,k} \rightarrow X_{t,k} .$$

To compose with  $(\lambda_{\mathbf{s}Z})_n$  is the same as to project over the components with  $s = 0$ , that is,  $t = l$  in the above equation. But the restriction of  $(\mu_Z)_n$  to these components is the identity.

Therefore  $\lambda_{\mathbf{s}X} \circ \mu_{\Delta \times X} = Id$ . The case  $Z = X \times \Delta$  is completely similar.

Finally, the associativity of  $\mu$  follows from the associativity of  $\mu_{E-Z}$  (proposition A.1.5), whereas the quasi-inverse of  $\lambda$  is just the inclusion of  $A$  as direct summand of  $\mathbf{s}(A \times \Delta)$ .

□

**REMARK 5.1.10.** As in the proof of the commutativity up to homotopy of diagram (5.4) in (SDC 7), it holds that given any chain complex  $A$ , there exists

a natural homotopy equivalence between  $cyl(A)$  and the classical cylinder of  $A$ . In addition this equivalence is compatible with the respective inclusions of  $A$  into both cylinders.

Hence, the properties deduced for  $Ch_*(\mathcal{A})$  from sections 2.2, 2.3 and 2.4, and from chapters 3 and 3, recover the classical treatment of the homotopy category associated with  $Ch_*(\mathcal{A})$ .

Another consequence is that each class  $E$  making  $Ch_*(\mathcal{A})$  into a simplicial descent category must contain the homotopy equivalences.

Again by (5.4), the shift functor induced by the descent structure coincide up to homotopy equivalence with the usual one. Thus it is an automorphism of categories  $HoCh_*(\mathcal{A}) \rightarrow HoCh_*(\mathcal{A})$ . Therefore, we obtain in this way the usual triangulated structure on  $HoCh_*(\mathcal{A})$  by theorem 4.1.17.

## 5.2 Chain complexes and quasi-isomorphisms

Now, let  $\mathcal{A}$  be an abelian category with numerable sums (or just an abelian category if we work in the uniformly bounded-below case, 5.2.4).

As in the additive case, consider the simple functor  $s = Tot \circ K : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$ , and the natural transformations  $\lambda$  and  $\mu$  given in definition 5.1.8.

As usual, a quasi-isomorphism is a morphism of  $Ch_*(\mathcal{A})$  that induces isomorphism in homology.

**PROPOSITION 5.2.1.** *Let  $\mathcal{A}$  be an abelian category with numerable sums. Then the category of chain complexes over  $\mathcal{A}$ , together with the quasi-isomorphisms as equivalences are an additive simplicial descent category. In addition  $\lambda$  is quasi-invertible and  $\mu$  is associative.*

*Proof.* Again, the two first axioms are well known properties. The axioms (SDC 3), (SDC 4), (SDC 5) and (SDC 8) follow directly from the additive case. The axiom (SDC 7) is again a consequence of 5.1.7 and of the commutativity up to homotopy equivalence of diagram 5.4 in the proof of (SDC 7) in proposition 5.1.9.

Finally, to see (SDC 6) it is enough to proof that if  $X \in \Delta^\circ Ch_*(\mathcal{A})$  is such that  $X_n$  is acyclic for all  $n \geq 0$ , then  $sX$  is so, by 2.1.11.

If  $X$  is such a simplicial chain complex, then  $KX$  is a double complex located in the right-half of the plane, and whose columns are acyclic. The following lemma state that  $Tot(KX) = sX$  is acyclic in this case.  $\square$



**LEMMA 5.2.2** ([B], p. 98, exercise 1.).

Let  $\{X_{p,q} ; d^1 : X_{p,q} \rightarrow X_{p-1,q} ; d^2 : X_{p,q} \rightarrow X_{p,q-1}\}$  be a double chain complex such that

1.  $X_{p,q} = 0$  if  $p < 0$ .

2. Given  $p \geq 0$ , the complex  $X_p = \{X_{p,q} ; d^2 : X_{p,q} \rightarrow X_{p,q-1}\}$  is acyclic.

Then  $Tot(X)$  is acyclic.

*Proof.* The proof given here is an adaptation of the same fact for “first quadrant” double complexes with acyclic columns.

Given  $q \geq 0$ , consider the subcomplex  $F^q \subseteq Tot(X)$  defined as

$$(F^l)_n = \bigoplus_{p+q=n ; p \leq l} X_{p,q}$$

that is in fact a subcomplex of  $Tot(X)$  since  $d^{Tot(X)}(F^q) \subseteq F^q$ .

In this way we obtain the increasing chain of subcomplexes of  $Tot(X)$

$$0 \subseteq X_0 = F^0 \subseteq \dots \subseteq F^q \subseteq F^{q+1} \subseteq \dots \subseteq Tot(X).$$

Moreover, we have the short exact sequence

$$0 \longrightarrow F^l \longrightarrow F^{l+1} \longrightarrow F^{l+1}/F^l \longrightarrow 0 .$$

Since  $F^{l+1}/F^l \simeq X_{l+1}$  and  $F^0 = X_0$  are acyclic, it follows by induction that  $F^q$  is acyclic for every  $q \geq 0$ .

If  $[x] \in H^r(Tot(X))$  is the class of  $x \in \ker\{d_r : Tot(X)_r \rightarrow Tot(X)_{r-1}\}$ , in particular  $x \in \bigoplus_{p+q=r} X_{p,q}$ . Then, by definition of direct sum, there exists a finite set of indexes  $I$  such that  $x \in \bigoplus_{(p,q) \in I} X_{p,q}$ . Therefore, we can find  $l$  such that  $x \in F_r^l$ . But  $d^{F_l} x = d_r^{Tot(X)} x = 0$  and  $F^l$  is acyclic, so  $x = d^{F^l} x' = d^{Tot(X)} x'$ . Hence  $[x] = 0$  and  $Tot(X)$  is acyclic.

In other words,  $Tot(X)$  is just the colimit of  $F^0 \subseteq \dots \subseteq F^q \subseteq F^{q+1} \subseteq \dots$ , where each  $F^q$  is acyclic, so  $Tot(X)$  is also acyclic since homology commutes with filtered colimits.  $\square$

As in the additive case, the shift functor induced by the descent structure coincide up to homotopy equivalence with the usual one. So it is an automorphism of  $HoCh_*(\mathcal{A})$ . Therefore, 4.1.17 recover the usual triangulated structure on the derived category of  $\mathcal{A}$ .

**REMARK 5.2.3.** In the abelian case, apart from the usual simple we can also consider the “normalized simple” that is defined using the normalized version of  $K$

$$K_N : \Delta^\circ \mathcal{B} \rightarrow Ch_* \mathcal{B}$$

instead  $K$ . Given a simplicial object  $B$  in  $\mathcal{B}$ ,  $K_N(B)$  is just the quotient of  $K(B)$  over the degenerate part of  $B$  [May]. Then,  $K_N$  provides the normalized simple functor

$$s_N : \Delta^\circ Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A}) ,$$

that also gives rise to a simplicial descent category structure on  $Ch_*(\mathcal{A})$ . This time  $\lambda = Id$ , whereas the transformation  $\mu$  of the non-normalized structure pass to quotient, inducing  $\mu_N : s_N \circ D \rightarrow s_N \Delta^\circ s_N$ .

This new descent structure is of course equivalent to the non-normalized one, since the projection  $K \rightarrow K_N$  is a homotopy equivalence [May], and applying  $Tot$  we obtain a homotopy equivalence relating  $s$  and  $s_N$ .

Consequently, the identity functor  $Ch_*(\mathcal{A}) \rightarrow Ch_*(\mathcal{A})$  is an equivalence of descent categories.

In the normalized case, the corresponding diagram (5.4) appearing in the proof of (SDC 7) is commutative. Similarly, if we compute  $cyl(A)$  using this normalized simple we obtain the usual cylinder associated with  $A$ , for each  $A$  in  $Ch_*(\mathcal{A})$ . In particular, the shift functor coincides with the usual one in this case.

**REMARK 5.2.4.** In sections 5.2 and 5.1 we have considered non bounded chain complexes, but all the properties contained in this section remain valid for uniformly bounded-bellow chain complexes.

Denote by  $Ch_q \mathcal{A}$  the category of chain complexes  $\{A_n, d\}$  with  $A_n = 0$  for all  $n$  smaller than the fixed bound  $q \in \mathbb{Z}$ . In this case, we don't need to impose the existence of numerable sums to define the simple functor, since we deal now with first-quadrant double complexes.

**PROPOSITION 5.2.5.** *Let  $\mathcal{A}$  be an additive (resp. abelian) category. Then  $Ch_q \mathcal{A}$  with the homotopy equivalences (resp. quasi-isomorphisms) as equivalences and the transformations  $\lambda$  and  $\mu$  given in 5.1.8, are an additive simplicial descent category. In addition  $\lambda$  is quasi-invertible and  $\mu$  is associative.*

In this case the shift functor is not an automorphism of  $HoCh_q \mathcal{A}$ , so  $HoCh_q \mathcal{A}$  is a suspended category (that is, right triangulated).

**REMARK 5.2.6.** The case  $Ch_{\mathbf{b}}\mathcal{A}$  of (non-uniformly) bounded-bellow chain complexes cannot be considered directly as an example of simplicial descent category, since the simple functor does not preserve bounded-bellow chain complexes.

However, we can use the previous proposition and argue as follows in order to give a proof based in these techniques of the well known triangulated structure on the derived category of  $Ch_{\mathbf{b}}\mathcal{A}$  (that is, the bounded-bellow derived category associated with  $\mathcal{A}$ ).

**COROLLARY 5.2.7.** *Let  $Ch_{\mathbf{b}}\mathcal{A}$  be the category of bounded-bellow chain complexes. Then the localized category  $HoCh_{\mathbf{b}}\mathcal{A}$  of  $Ch_{\mathbf{b}}\mathcal{A}$  with respect to the quasi-isomorphisms (resp. the homotopy equivalences) is a triangulated category.*

*Proof.* Let us prove the case  $HoCh_{\mathbf{b}}\mathcal{A} = D_{\mathbf{b}}\mathcal{A}$ , the localized category of  $Ch_{\mathbf{b}}\mathcal{A}$  with respect to the quasi-isomorphisms. The other case is completely similar. The idea of the proof is just to induce in  $D_{\mathbf{b}}\mathcal{A} = \bigcup_{k \in \mathbb{Z}} HoCh_k\mathcal{A}$  the suspended category structure coming from each  $HoCh_k\mathcal{A}$ , and since in  $D_{\mathbf{b}}\mathcal{A}$  the shift functor is an automorphism, it follows that  $D_{\mathbf{b}}\mathcal{A}$  is triangulated.

Before localizing, we have the chain of inclusions of categories

$$\cdots \subset Ch_k\mathcal{A} \subset Ch_{k+1}\mathcal{A} \subset \cdots \subset Ch_{\mathbf{b}}\mathcal{A}$$

and  $Ch_{\mathbf{b}}\mathcal{A} = \bigcup_{k \in \mathbb{Z}} Ch_k\mathcal{A}$ .

For any  $k \in \mathbb{Z}$ , the category  $Ch_k\mathcal{A}$  is an additive simplicial descent category. In particular, we have the cone functor  $c^k : Fl(Ch_k\mathcal{A}) \rightarrow Ch_k\mathcal{A}$ , that is compatible with the inclusions  $Ch_k\mathcal{A} \subset Ch_{k+1}\mathcal{A}$ . This compatibility holds because the simple functor does not depend on  $k$ . Then, the family  $\{c^k\}$  induces the cone functor  $c : Fl(Ch_{\mathbf{b}}\mathcal{A}) \rightarrow Ch_{\mathbf{b}}\mathcal{A}$ , that is well defined. Therefore, the shift functor  $[1] : Ch_{\mathbf{b}}\mathcal{A} \rightarrow Ch_{\mathbf{b}}\mathcal{A}$  is also defined. Moreover, it preserves quasi-isomorphisms, so it passes to the derived categories.

Given a morphism  $f : X \rightarrow Y$  in  $Ch_{\mathbf{b}}\mathcal{A}$ , there exists  $K \in \mathbb{Z}$  such that  $f$  is in  $Ch_K\mathcal{A}$ , so  $f$  gives rise to the triangle

$$X \xrightarrow{f} Y \longrightarrow c(f) \longrightarrow X[1] \quad (5.5)$$

where all arrows are in  $Ch_K\mathcal{A}$  because  $\lambda$  is quasi-invertible.

Define the class of distinguished triangles of  $D_{\mathbf{b}}\mathcal{A}$  as those isomorphic (in  $D_{\mathbf{b}}\mathcal{A}$ ) to some triangle in the form (5.5).

This class of distinguished triangles is by definition closed by isomorphism. Each distinguished triangle of  $D_{\mathbf{b}}\mathcal{A}$  is isomorphic to some distinguished triangle

of  $D_K\mathcal{A}$ , for some  $K$ . Since  $D_K\mathcal{A}$  is suspended (theorem 4.1.17). Thus the distinguished triangles of  $D_{\mathbf{b}}\mathcal{A}$  satisfy all axioms of suspended categories.

Consequently,  $D_{\mathbf{b}}\mathcal{A}$  is suspended, and since  $[1]$  is an automorphism, it is triangulated.  $\square$

**REMARK 5.2.8.** The above proof can be generalized to the case in which a category  $\mathcal{D}$  is the inductive limit of a family of simplicial descent categories, because the argument used above just means that theorem 4.1.17 is preserved by “inductive limits”.

### 5.3 Simplicial objects in additive or abelian categories

The Eilenberg-Zilber-Cartier theorem A.1.1 admits an interesting interpretation in the context of descent categories. Under our setting, this theorem means that the simplicial objects in an additive category are a simplicial descent category, taking as simple the diagonal functor, and as equivalences the morphisms that are mapped into homotopy equivalences in  $Ch_*\mathcal{A}$ .

Let  $Ch_+\mathcal{A}$  be the category of positive chain complexes of  $\mathcal{A}$  (that is,  $Ch_+\mathcal{A} = Ch_0\mathcal{A}$  in 5.2.4).

Remind that the functor  $K : \Delta^\circ\mathcal{A} \rightarrow Ch_+\mathcal{A}$  given in 5.1.8 is defined by taking as boundary map the alternate sum of the face maps.

Next definition describes the descent structure on  $\Delta^\circ\mathcal{A}$ .

#### DEFINITION 5.3.1.

**Simple functor:** The simple functor is the diagonal functor  $D : \Delta^\circ\Delta^\circ\mathcal{A} \rightarrow \Delta^\circ\mathcal{A}$ .

**Equivalences:** The equivalences are the class

$$E = \{f \in \Delta^\circ\mathcal{A} \mid K(f) \text{ is a homotopy equivalence}\}.$$

**Transformations  $\lambda$  and  $\mu$ :** The natural transformations  $\lambda$  and  $\mu$  are defined as the corresponding identity natural transformation.

**PROPOSITION 5.3.2.** *Under the previous notations,  $(\Delta^\circ\mathcal{A}, E, D, \mu, \lambda)$  is an additive simplicial descent category, in which  $\lambda$  is quasi-invertible and  $\mu$  associative.*

*In addition,  $K : \Delta^\circ\mathcal{A} \rightarrow Ch_+\mathcal{A}$  is a functor of additive simplicial descent categories, where we consider in  $Ch_+\mathcal{A}$  the descent structure given in proposition 5.2.5.*

*Proof.* Let us check that  $K : \Delta^\circ \mathcal{A} \rightarrow Ch_+ \mathcal{A}$  satisfies the hypothesis of transfer lemma 2.5.8.

The axioms (SDC 1) and (SDC 3)' are clear. Let us see (SDC 4)'.

Since  $\mathbf{s} = \mathbf{D}$ , the compositions  $\mathbf{s}\Delta^\circ \mathbf{s}$  and  $\mathbf{s}\mathbf{D}$  are equal, so it suffices to take  $\mu = Id : \mathbf{s}\mathbf{D} \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}$ .

To see (SDC 5)', consider  $X \in \Delta^\circ \Delta^\circ \mathcal{A}$ . Then  $\mathbf{D}(X \times \Delta) = X$ , and we can set  $\lambda = Id$ .

The compatibility between  $\lambda$  and  $\mu$  holds trivially. In addition,  $K$  is an additive functor, in particular it is quasi-strict monoidal, so (FD 1) holds.

It remains to see (FD 2). Denote by  $(Ch_+ \mathcal{A}, E', \mathbf{s}', \mu', \lambda')$  the descent structure on  $Ch_+ \mathcal{A}$  given in 5.2.5. By theorem A.1.1 a) for  $\mathcal{U} = \mathcal{A}$ , we deduce that  $\Theta = \mu_{E-Z} : K \circ \mathbf{D} \rightarrow \mathbf{s}' \circ \Delta^\circ K$  is a homotopy equivalence when we evaluate it at each bisimplicial object of  $\mathcal{A}$ .

Then, if  $X \in \Delta^\circ \Delta^\circ \mathcal{A}$ , the morphism  $\Theta_X = \mu_{E-Z} : K\mathbf{D}(X) \rightarrow \mathbf{s}'\Delta^\circ K(X)$  is “universal” and such that  $(\Theta_X)_0 = Id_{X_{0,0}} : X_{0,0} \rightarrow X_{0,0}$ .

Let us check the compatibility between  $\lambda$ ,  $\mu$ ,  $\Theta$ ,  $\lambda'$  and  $\mu'$ .

Given  $X \in \Delta^\circ \mathcal{A}$ , denote by  $\tilde{X}$  the associated constant simplicial object, that is  $\tilde{X}_{n,m} = X_m$  for all  $n, m$ . We must see that  $\lambda'_{KX} \circ \mu_{E-Z}(\tilde{X}) = Id_{KX}$  in  $Ch_+ \mathcal{A}$ .

By definition,  $(\lambda'_{KX})_n : X_n \oplus \cdots \oplus X_0 \rightarrow X_n$  is the projection, whereas  $\mu_{E-Z}(\tilde{X}) : X_n \rightarrow X_n \oplus \cdots \oplus X_0$  is  $\mu_{E_Z}(\tilde{X}) = (Id, d_n, d_{n-1} \circ d_n, \dots, d_1 \circ \cdots \circ d_n)$ . Therefore, when we project over the component  $X_n$ , we obtain the identity.

The compatibility between  $\Theta = \mu_{E_Z}$  and  $\mu'$  (also obtained from  $\mu_{E-Z}$ ) is consequence of the associativity of this transformation A.1.5.

Finally,  $\Delta^\circ \mathcal{A}$  is additive since  $\mathcal{A}$  is, and the diagonal functor  $\mathbf{D} : \Delta^\circ \Delta^\circ \mathcal{A} \rightarrow \mathcal{A}$  is additive, so  $\Delta^\circ \mathcal{A}$  is an additive simplicial descent category.  $\square$

Assume now that  $\mathcal{A}$  is an abelian category, and  $K : \Delta^\circ \mathcal{A} \rightarrow Ch_+ \mathcal{A}$  the usual functor. We have the additional descent structure on  $\Delta^\circ \mathcal{A}$ .

### DEFINITION 5.3.3.

**Simple functor:** Again, the simple functor is the diagonal functor  $\mathbf{D} : \Delta^\circ \Delta^\circ \mathcal{A} \rightarrow \Delta^\circ \mathcal{A}$ .

**Equivalences:** The class of equivalences is

$$E' = \{f \in \Delta^\circ \mathcal{A} \mid K(f) \text{ is a quasi-isomorphism in } Ch_+ \mathcal{A}\}.$$

**Transformations  $\lambda$  and  $\mu$ :** The natural transformations  $\lambda$  and  $\mu$  are defined as the identity natural transformation.

**PROPOSITION 5.3.4.** *Under the above notations,  $(\Delta^\circ \mathcal{A}, E', D, \mu, \lambda)$  is an additive simplicial descent category such that  $\lambda$  is quasi-invertible and  $\mu$  is associative.*

*In addition,  $K : \Delta^\circ \mathcal{A} \rightarrow Ch_+ \mathcal{A}$  is a functor of additive simplicial descent categories, where the descent structure on  $Ch_+ \mathcal{A}$  is the one given in proposition 5.2.5.*

*Proof.* From the proof of proposition 5.3.2 we deduce that  $K : \Delta^\circ \mathcal{A} \rightarrow Ch_+ \mathcal{A}$  satisfies the conditions of the transfer lemma 2.5.8, now taking the descent structure on  $Ch_+ \mathcal{A}$  in with the equivalences are the quasi-isomorphisms 5.2.5.  $\square$

## 5.4 Simplicial Sets

Denote by *Set* the category of sets, and by *Ab* the category of abelian groups.

In this section we will give a descent structure to  $\Delta^\circ Set$ , in which the equivalences will be the quasi-isomorphisms.

**DEFINITION 5.4.1.** If  $L : Set \rightarrow Ab$  is the functor that maps a set  $T$  to the free group with base  $T$ , then the homology of a simplicial set  $W$  is the homology of the chain complex  $K \circ \Delta^\circ L(W)$ , that is the image of  $W$  under the composition of functors

$$\Delta^\circ Set \xrightarrow{\Delta^\circ L} \Delta^\circ Ab \xrightarrow{K} Ch_+(Ab).$$

**DEFINITION 5.4.2.**

**Simple functor:** Again, the simple functor is the diagonal functor  $D : \Delta^\circ \Delta^\circ Set \rightarrow \Delta^\circ Set$ .

**Equivalences:** The class  $E$  of equivalences consists of those morphisms that induce isomorphism in homology.

**Transformations  $\lambda$  and  $\mu$ :** The natural transformations  $\lambda$  and  $\mu$  are defined as the identity natural transformation.

**PROPOSITION 5.4.3.** *Under the above notations,  $(\Delta^\circ Set, E, D, \mu, \lambda)$  is a simplicial descent category such that  $\lambda$  is quasi-invertible and  $\mu$  is associative.*

*In addition,  $\Delta^\circ L : \Delta^\circ Set \rightarrow \Delta^\circ Ab$  is a functor of simplicial descent categories, where the descent structure on  $\Delta^\circ Ab$  is the one given in proposition 5.3.4.*

*Proof.* The compatibility between  $\lambda$  and  $\mu$  is clear. The functor  $\Delta^\circ L : \Delta^\circ Set \rightarrow \Delta^\circ Ab$  satisfies the hypothesis of the transfer lemma 2.5.8 trivially, where  $\Theta$  is again the identity natural transformation.  $\square$

**REMARK 5.4.4.** The homology of a simplicial set  $W$  coincides with the singular homology of its geometric realization  $|W|$  (see [May] 16.2 ii)). Then

$$E = \{f : W \rightarrow W' \mid |f| : |W| \rightarrow |L| \text{ induces isomorphism in singular homology}\}$$

## 5.5 Topological Spaces

Consider the category  $Top$  of topological spaces and continuous maps. We will endow the category  $Top$  with a descent structure in which the equivalences are the quasi-isomorphisms (that is, morphisms inducing isomorphism in singular homology).

The usual geometric realization  $|\cdot| : \Delta^\circ Top \rightarrow Top$  present some disadvantages for our purposes. For instance, we need to impose some extra conditions to a map  $f : X \rightarrow Y$  such that  $f_n$  is an equivalence for all  $n$  in order to have that  $|f|$  is again an equivalence (see, for instance, [M] 11.13). In other words, the exactness axiom of simplicial descent categories is not satisfied by  $|\cdot|$  under the generality needed here.

This is the reason why we consider as simple the so called “fat” geometric realization, defined in a similar way as  $|\cdot|$ , except that now we do not identify those terms related through the degeneracy maps (we only identify those terms related through the face maps).

The natural transformation  $s \rightarrow |\cdot|$  is a homotopy equivalence when evaluated at those  $X \in \Delta^\circ Top$  such that the degeneracy maps are closed cofibrations (see [S], appendix A), for instance when evaluated at simplicial sets.

**DEFINITION 5.5.1.** Let  $\Delta : \Delta \rightarrow Top$  be the functor which maps the ordinal  $[m]$  in  $\Delta$  to the standard  $m$ -dimensional simplex  $\Delta^m \subset \mathbb{R}^{m+1}$  given by

$$\Delta^m = \{(t_0, \dots, t_m) \in \mathbb{R}^{m+1} \mid \sum_{k=0}^m t_k = 1 \text{ and } t_k \geq 0\}.$$

If  $f : [n] \rightarrow [m]$  is a morphism of  $\Delta$ , then  $f$  induces a continuous map  $\Delta(f) : \Delta^n \rightarrow \Delta^m$ .

Setting  $J_i = f^{-1}(\{i\})$ , then  $\Delta(f)(t_0, \dots, t_n) = (r_1, \dots, r_m)$  where  $r_i = \sum_{j \in J_i} t_j$  if  $J_i$  is not empty, and  $r_i = 0$  otherwise.

Recall that the singular homology of a topological space is by definition the homology of the simplicial set obtained through the “singular chains” functor. This functor  $S : Top \rightarrow \Delta^\circ Set$  assigns to a topological space  $X$  the simplicial set

$$SX = \{Hom_{Top}(\Delta^n, X)\}_n.$$

Then, the singular homology of  $X$  is just the homology of the chain complex  $K\Delta^\circ L(SX)$  given in definition 5.4.1.

**DEFINITION 5.5.2.**

**Simple functor:** the simple functor  $\mathbf{s} : \Delta^\circ Top \rightarrow Top$  is the “fat” geometric realization. Given a simplicial topological space

$$\begin{array}{ccccccc} X_0 & \xleftarrow{\partial_1} & X_1 & \xleftarrow{\partial_1} & X_2 & \xleftarrow{\partial_1} & X_3 & \cdots \\ & \searrow \partial_0 & \swarrow \partial_0 & \searrow \partial_0 & \swarrow \partial_0 & \searrow \partial_0 & \swarrow \partial_0 & \\ & s_0 & & & & & & \end{array}$$

$$\begin{array}{ccc} \Delta_e^\circ \times \Delta_e & \longrightarrow & Top \\ ([n], [m]) & \longrightarrow & X_n \times \Delta^m \end{array}$$

The fat geometric realization of  $X$  is defined as the cofinal of this bifunctor (cf. [ML]):

$$\mathbf{s}X = \int^n X_n \times \Delta^n$$

more specifically,

$$\mathbf{s}X = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where  $\sim$  is the equivalence relation generated by

$$(\partial_i(x), u) \sim (x, \Delta(d_i)(u)) \text{ if } d_i : [n-1] \rightarrow [n], \text{ and } (x, u) \in X_n \times \Delta^{n-1}.$$

We will write  $[x, t]$  for the equivalence class of an element  $(x, t) \in \coprod_{n \geq 0} X_n \times \Delta^n$ .

**Equivalences:** Consider the class  $E$  consisting of those continuous maps that induce isomorphism in singular homology. This kind of maps will be called quasi-isomorphisms as well.

**Transformation  $\lambda$ :** Given  $X \in Top$ , the projections  $p_n : X \times \Delta^n \rightarrow X$  induce by the universal property of cofinals a continuous map  $\lambda_X : \mathbf{s}(X \times \Delta) \rightarrow X$ , natural in  $X$ , with  $\lambda_X[x, t] = x$ .

**Transformation  $\mu$ :** If  $Z \in \Delta^\circ \Delta^\circ Top$ , since the simple functor is defined as a cofinal, the Fubini theorem holds ([ML], IX.8), and we deduce that  $\mathbf{s} \Delta^\circ \mathbf{s}Z$  is the quotient of  $\coprod_{n, m \geq 0} Z_{n, m} \times \Delta^n \times \Delta^m$  over the obvious identifications.

Then, the maps  $(\mu_Z)_n : Z_{n, n} \times \Delta^n \rightarrow \mathbf{s} \Delta^\circ \mathbf{s}Z$  with  $(\mu_Z)_n[z, t] = [z, t, t]$  provides a continuous map  $\mu_Z : \mathbf{s}DZ \rightarrow \mathbf{s} \Delta^\circ \mathbf{s}Z$  such that  $\mu_Z([z_{nn}, t_n]) = [z_{n, n}, t_n, t_n]$ .

**PROPOSITION 5.5.3.** *Under the previous notations,  $(Top, E, \mathbf{s}, \mu, \lambda)$  is a simplicial descent category, such that  $\mu$  is associative and  $\lambda$  is quasi-invertible.*



In addition, the singular functor  $S : Top \rightarrow \Delta^\circ Set$  is a functor of simplicial descent categories.

Now we will begin with the proof of this proposition. To this end we need some preliminary results.

**LEMMA 5.5.4.** *If  $f : X \rightarrow Y$  is a morphism in  $\Delta^\circ Top$  such that for all  $n$ ,  $f_n$  induces isomorphism in singular homology (resp.  $f_n$  is a homotopy equivalence) then the same holds for  $\mathbf{s}f : \mathbf{s}X \rightarrow \mathbf{s}Y$ .*

The proof for  $f_n$  quasi-isomorphism can be found in [Dup] 5.16, and the case  $f_n$  homotopy equivalence appears in [S] A.1.

**REMARK 5.5.5.** The previous lemma justifies the choice of  $\mathbf{s}$  as simple functor instead of the usual geometric realization  $|\cdot|$ . On the other hand, one of the advantages of  $|\cdot|$  is the existence of the adjoint pair ([May], §16)

$$\Delta^\circ Set \xrightleftharpoons[\mathbf{s}]{|\cdot|} Top . \quad (5.6)$$

Our simple functor  $\mathbf{s}$  is defined by forgetting the degeneracy maps of a simplicial topological space. A consequence of this fact is that the above adjunction does not hold at the level of simplicial sets between  $\mathbf{s}$  and  $S$ , but it holds at the level of strict simplicial sets. That is to say, there is an adjunction

$$\Delta_e^\circ Set \xrightleftharpoons[\mathbf{s}]{\mathbf{s}} Top .$$

Due to this fact we will have to solve some technical difficulties in the proof of proposition 5.5.3.

**REMARK 5.5.6.** Note that we can consider as well the homology of a strict simplicial set  $W \in \Delta_e^\circ Set$ , because the functor  $K$  does not use the degeneracy maps of a simplicial set, that is,  $K : \Delta_e^\circ Ab \rightarrow Ch_+(Ab)$ .

Then, quasi-isomorphisms between strict simplicial sets can be defined in the same way as those morphisms  $f : W \rightarrow W'$  in  $\Delta_e^\circ Set$  that induce isomorphisms in homology.

**DEFINITION 5.5.7.** The geometric realization  $|\cdot| : \Delta^\circ Top \rightarrow Top$  is defined as

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where  $\sim$  is the equivalence relation generated by

$$(\theta(x), u) \sim (x, \Delta(\theta)(u)) \text{ if } \theta : [m] \rightarrow [n] \text{ is a morphisms of } \Delta, \text{ and } (x, u) \in X_n \times \Delta^m.$$

We will write  $|x, t|$  for the equivalence class of the element  $(x, t) \in \coprod_{n \geq 0} X_n \times \Delta^n$ .

**DEFINITION 5.5.8.** A simplicial topological space is called “good” if its degeneracy maps are closed cofibrations.

Simplicial sets (with the discrete topology) are always good in this sense.

Moreover, if  $T \in \Delta^\circ \Delta^\circ \text{Set}$  is a bisimplicial set, then the simplicial topological space  $X$  with  $X_n = |m \rightarrow T_{m,n}|$ , obtained by applying the geometric realization to  $T$  with respect to one of its indexes is also an example of “good” simplicial topological space.

This is consequence of the fact that any degeneracy map  $s_j : T_{\cdot,n} \rightarrow T_{\cdot,n+1}$  is an inclusion of simplicial sets (because of the simplicial identities), and the geometric realization of any inclusion of simplicial sets is always a closed cofibration.

Next we recall the following connection between  $|\cdot|$  and  $\mathbf{s}$ , given in [S], appendix A.

**LEMMA 5.5.9.** *If  $X$  is a good simplicial topological space then the morphism*

$$\tau_X : \mathbf{s}X \rightarrow |X| \quad [x, t] \rightarrow |x, t|$$

*is a homotopy equivalence.*

The fat geometric realization satisfies as well the classical Eilenberg-Zilber property.

**LEMMA 5.5.10** (Eilenberg-Zilber). *Given  $W \in \Delta^\circ \Delta^\circ \text{Set}$ , the map  $\eta(W) : |D(W)| \rightarrow |\Delta^\circ W|$ , with  $\eta(W)([w_{nn}, t_n]) = |w_{n,n}, t_n, t_n|$  is an homeomorphism. The map  $\mu(W) : \mathbf{s}(D(W)) \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}(W)$ , with  $\mu(W)([w_{nn}, t_n]) = [w_{n,n}, t_n, t_n]$  is a homotopy equivalence. In addition, the following diagram commutes*

$$\begin{array}{ccc} \mathbf{s}D(W) & \xrightarrow{\mu(W)} & \mathbf{s}\Delta^\circ \mathbf{s}(W) \\ \downarrow \tau & & \downarrow P \\ |D(W)| & \xrightarrow{\eta(W)} & |\Delta^\circ W| \end{array} \quad (5.7)$$

where  $P : \mathbf{s}\Delta^\circ \mathbf{s}(W) \rightarrow |\Delta^\circ W|$ ,  $[x, p, q] \rightarrow |x, p, q|$ .

*Proof.* Firstly, the usual Eilenberg-Zilber theorem ([GM]. I.3.7) states that  $\eta(W) : |D(W)| \rightarrow |\Delta^\circ W|$  is a homeomorphism. The second part of the lemma is a consequence of the commutativity of diagram (5.7), since both  $\tau$

(lemma 5.5.9) and  $P$  are homotopy equivalences.

Fix  $n$  as the first index of  $W$ , obtaining  $W_{n,\cdot} \in \Delta^\circ \text{Set}$ . The projections  $p_n = \tau_{W_n} : \mathbf{s}(W_{n,\cdot}) \rightarrow |W_{n,\cdot}|$  are homotopy equivalences for all  $n$  by 5.5.9.

Hence,  $p_\cdot = \{p_n\}_n : \Delta^\circ \mathbf{s}(W) \rightarrow \Delta^\circ |W|$  is such that  $\mathbf{s}(p_\cdot) : \mathbf{s}\Delta^\circ \mathbf{s}(W) \rightarrow \mathbf{s}\Delta^\circ |W|$  is a homotopy equivalence by 5.5.4.

Since  $\Delta^\circ |W|$  is a good simplicial topological space then the projection  $\mathbf{s}\Delta^\circ |W| \rightarrow |\Delta^\circ |W||$  is also a homotopy equivalence, and composing both maps we deduce that the projection  $P : \mathbf{s}\Delta^\circ \mathbf{s}(W) \rightarrow |\Delta^\circ |W||$  is a homology equivalence as required.  $\square$

The following statement is similar to the classical result satisfied by the usual geometric realization  $|\cdot|$ .

**LEMMA 5.5.11.** *Consider the natural transformations  $\Phi : \mathbf{s}S \Rightarrow Id_{Top}$  and  $\Psi : Id_{\Delta_e^\circ \text{Set}} \Rightarrow S\mathbf{s}$  defined as  $\Phi(Y)([\lambda_n, t_n]) = \lambda_n(t_n)$ ,  $(\Psi(W)_n(w_n))(t_n) = [w_n, t_n]$ . Then  $\Phi$  and  $\Psi$  induce isomorphism in homology for all  $Y \in Top$  and for all  $W \in \Delta_e^\circ \text{Set}$ .*

*Proof.* As stated in the proof of ([Dup], 5.15), given  $W \in \Delta_e^\circ \text{Set}$ , then the fat geometric realization of  $W$ ,  $\mathbf{s}W$ , is a CW-complex whose  $n$ -cell are the set  $W_n$ . Then, the group of  $n$ -cellular chains of  $\mathbf{s}W$  is just  $L(W_n)$  and the cellular boundary map is  $\sum_i (-1)^i d_i$ .

Since the cellular homology and the singular homology of a CW-complex coincide, then the morphism  $\Psi(W)$  of  $\Delta_e^\circ \text{Set}$  induces isomorphism in homology.

Consider now  $Y \in Top$ . By the first part of the lemma,  $\Psi(SY) : SY \rightarrow S\mathbf{s}(SY)$  induces isomorphism in homology. But  $S\Phi(Y) \circ \Psi S(Y) = Id : SY \rightarrow SY$ , so  $S\Phi(Y)$  is also a quasi-isomorphism. Hence  $\Phi(Y)$  induces isomorphism in singular homology.  $\square$

We need use the next technical result.

**LEMMA 5.5.12.** *Let  $Ho\Delta^\circ \text{Set}$  (resp.  $Ho\Delta_e^\circ \text{Set}$ ) be the localized category of simplicial sets (resp. strict simplicial sets) with respect to the quasi-isomorphisms. Then, the forgetful functor  $U : \Delta^\circ \text{Set} \rightarrow \Delta_e^\circ \text{Set}$  preserves quasi-isomorphisms, giving rise to the functor*

$$U : Ho\Delta^\circ \text{Set} \rightarrow Ho\Delta_e^\circ \text{Set} .$$

*This is a faithful functor, that is, the map  $Hom_{Ho\Delta^\circ \text{Set}}(W, L) \rightarrow Hom_{Ho\Delta_e^\circ \text{Set}}(UW, UL)$  is injective.*

We will delay the proof of this lemma to give before the one of proposition 5.5.3.

**Proof of 5.5.3.**

Let us check that the hypothesis of proposition 2.5.8 are satisfied by the singular chain complex functor  $S : Top \rightarrow \Delta^\circ Set$ .

First, note that the transformations  $\lambda$  and  $\mu$  are compatible. To see this, let  $X$  be a simplicial topological space, and  $[x, t]$  an element of  $\mathbf{s}X$  representing the pair  $(x, t) \in X_n \times \Delta^n$ .

Then  $\lambda_{\mathbf{s}X} \circ \mu_{\Delta \times X}([x, t]) = \lambda_{\mathbf{s}X}([x, t], t) = [x, t]$ , so  $\lambda_{\mathbf{s}X} \circ \mu_{\Delta \times X} = Id$ , and similarly  $\mathbf{s}(\lambda_{X_n}) \circ \mu_{X \times \Delta} = Id$ .

The disjoint union is the coproduct in  $Top$ , and the singular chain functor commutes with coproducts, so (FD 1) holds.

In order to relax the notations, we will write also  $\psi$  for the induced functors  $\Delta^\circ \psi : \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \mathcal{D}'$  and  $\Delta^\circ \Delta^\circ \psi : \Delta^\circ \Delta^\circ \mathcal{D} \rightarrow \Delta^\circ \Delta^\circ \mathcal{D}'$ .

To see (FD 2), we must study the commutativity of the diagram

$$\begin{array}{ccc} \Delta^\circ Top & \xrightarrow{S} & \Delta^\circ \Delta^\circ Set \\ \downarrow \mathbf{s} & & \downarrow D \\ Top & \xrightarrow{S} & \Delta^\circ Set . \end{array}$$

Define the isomorphism  $\Theta_X : S(\mathbf{s}X) \rightarrow D(SX)$  of  $Ho\Delta^\circ Set$  as the one coming from the zig-zag in  $\Delta^\circ Set$

$$S(\mathbf{s}X) \xleftarrow{\Theta_X^0} S(\mathbf{s}\Delta^\circ \mathbf{s}(SX)) \xrightarrow{\Theta_X^1} S|\Delta^\circ|SX| \xleftarrow{\Theta_X^2} D(SX)$$

defined as follows.

The transformation  $\Theta_X^0 : S(\mathbf{s}\Delta^\circ \mathbf{s}(SX)) \rightarrow S(\mathbf{s}X)$  is just the image under  $S\mathbf{s}$  of the morphism  $\phi : \Delta^\circ \mathbf{s}(SX) \rightarrow X$  of  $\Delta^\circ Top$ , which in degree  $n$  is given by  $\Phi(X_n) : \mathbf{s}SX_n \rightarrow X_n$  (see 5.5.11). Therefore, if  $\alpha : \Delta^n \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}(SX)$  is the morphism that assigns to  $t \in \Delta^n$  the class of  $(\beta, p, q) \in S_k X_m \times \Delta^k \times \Delta^m$ , it follows that

$$\Theta_X^0(\alpha)(t) = \mathbf{s}\phi \circ \alpha(t) = [\beta(p), q] \in \mathbf{s}X .$$

Then, as each  $\Phi(X_n)$  is a quasi-isomorphism, we deduce from 5.5.4 that  $\mathbf{s}\phi$  (and hence  $S\mathbf{s}\phi = \Theta_X^0$ ) is so.

The transformation  $\Theta_X^1 : S(\mathbf{s}\Delta^\circ \mathbf{s}(SX)) \rightarrow S|\Delta^\circ|SX|$  is the image under  $S$  of the projection  $P : \mathbf{s}\Delta^\circ \mathbf{s}(SX) \rightarrow S|\Delta^\circ|SX|$ ,  $P([x, t, r]) = [x, t, r]$ . In 5.5.10 we checked that  $P$  is a homotopy equivalence for any bisimplicial set  $W$ , in

particular it is so for  $W = SX$ .

Secondly,  $\Theta_X^2 : D(SX) \rightarrow S|\Delta^\circ|SX||$  assigns to  $\alpha : \Delta^n \rightarrow X_n$  the morphism  $\Theta_X^2(\alpha) : \Delta^n \rightarrow |\Delta^\circ|SX||$  given by  $t \rightarrow |\alpha, t, t|$ , where  $(\alpha, t, t) \in S_n X_n \times \Delta^n \times \Delta^n$ .

Note that  $\Theta_X^2$  induces isomorphism in homology, since it is just the composition

$$\begin{array}{ccccc} D(SX) & \xrightarrow{\Psi'(SX)} & S|D(SX)| & \xrightarrow{S(\eta(SX))} & S|\Delta^\circ|SX|| \\ \alpha : \Delta^n \rightarrow X_n & \longrightarrow & t \in \Delta^n \rightarrow |\alpha, t| & \longrightarrow & t \in \Delta^n \rightarrow |\alpha, t, t| \end{array}$$

The morphism  $\Psi'(SX)$  comes from the adjunction (5.6) and it is a quasi-isomorphism (cf. [May] 16.2), whereas  $\eta(SX)$  is a homeomorphism by 5.5.10, so  $S(\eta(SX))$  is an isomorphism of simplicial sets.

It remains to prove that  $\Theta$  is compatible with the natural transformations  $\lambda$  and  $\mu$  of  $Top$  and  $\Delta^\circ Set$ . To see this, by lemma 5.5.12 it suffices to check that the corresponding diagrams commute in  $Ho\Delta_e^\circ Set$ . The advantage of wording in  $\Delta_e^\circ Set$  is that  $U\Theta$  coincides in  $Ho\Delta_e^\circ Set$  with the morphism of strict simplicial sets

$$S(\mathbf{s}X) \xleftarrow{\theta(X)} D(SX)$$

such that  $\theta(X)_n : S_n X_n = \{\gamma : \Delta^n \rightarrow X_n\} \rightarrow S\mathbf{s}(X) = \{\zeta : \Delta^n \rightarrow \mathbf{s}(X)\}$  is  $\theta(X)_n(\gamma)(t) = [\gamma(t), t] \in \mathbf{s}(X)$ .

Indeed, the morphism  $\theta'(X)$  of  $\Delta_e^\circ Set$  defined as

$$\begin{array}{ccc} D(SX) & \xrightarrow{\theta'(X)} & S(\mathbf{s}\Delta^\circ \mathbf{s}(SX)) \\ \alpha : \Delta^n \rightarrow X_n & \longrightarrow & t \in \Delta^n \rightarrow [\alpha, t, t] \end{array}$$

fits into the commutative diagram

$$\begin{array}{ccc} S(\mathbf{s}\Delta^\circ \mathbf{s}(SX)) & \xleftarrow{\theta'(X)} & D(SX) \\ & \searrow U\Theta_X^1 & \swarrow U\Theta_X^2 \\ & S|\Delta^\circ|SX|| & \end{array} .$$

Hence  $U\Theta = (U\Theta_X^1 \circ \theta'(X))^{-1} = \theta(X)^{-1} : S(\mathbf{s}X) \rightarrow D(SX)$ .

Firstly, given  $Y \in Top$ , diagram (2.18) in  $Ho\Delta_e^\circ Set$  is now

$$\begin{array}{ccc} S(\mathbf{s}(Y \times \Delta)) & & \\ \uparrow \theta(Y \times \Delta) & \searrow S(\lambda_Y) & \\ D((SY) \times \Delta) & \xrightarrow{Id} & SY . \end{array}$$

whose commutativity in  $\Delta_e^{\circ}Set$  follows directly from the definitions.

On the other hand, given  $Z \in \Delta^{\circ}\Delta^{\circ}Top$ , (2.18) is now the following diagram of  $Ho\Delta_e^{\circ}Set$

$$\begin{array}{ccc} DD(SZ) & \xrightarrow{Id} & DD(SZ) \\ \theta(DZ) \downarrow & & \downarrow D\theta(Z) \\ & & DS(\Delta^{\circ}sZ) \\ & & \downarrow \theta(\Delta^{\circ}sZ) \\ Ss(DZ) & \xrightarrow{S\mu_Z} & Ss(\Delta^{\circ}sZ) \end{array}$$

whose commutativity is again a direct consequence of definitions.

Therefore, the transfer lemma is proven for  $S : Top \rightarrow \Delta^{\circ}Set$ .

To finish the proof,  $\mu$  is clearly associative, whereas the quasi-inverse of  $\lambda$ ,  $\lambda' : Id_{Top} \rightarrow s(- \times \Delta)$ , can be defined as follows. If  $X \in Top$  and  $x \in X$ , then  $\lambda'_X(x)$  is the equivalence class in  $s(X \times \Delta)$  of the pair  $(x, *) \in X \times \Delta^0$ .  $\square$

**Proof of lemma 5.5.12.** The proof is based in the properties satisfied by the adjoint pair

$$\Delta_e^{\circ}Set \xrightleftharpoons[\text{U}]{\pi} \Delta^{\circ}Set .$$

where  $\pi : \Delta_e^{\circ}Set \rightarrow \Delta^{\circ}Set$  is the Dold-Puppe transform (see 1.1.16).

*Step 1:* Denote also by  $\pi : \Delta_e^{\circ}Ab \rightarrow \Delta^{\circ}Ab$  to the Dold-Puppe transform in the category of abelian groups. We will use the functors  $K : \Delta_e^{\circ}Ab \rightarrow Ch_+Ab$ ,  $K : \Delta^{\circ}Ab \rightarrow Ch_+Ab$  and  $K^N : \Delta^{\circ}Ab \rightarrow Ch_+Ab$ , where  $K$  is as usual the functor that takes the alternate sums of face maps as boundary map, whereas

$$K^N(A) = K(A)/D(A), \text{ where } D(A)_n = \bigcup_{i=0}^{n-1} s_i A_{n-1} .$$

Given any  $W \in \Delta_e^{\circ}Ab$ , we will see that

$$K^N(\pi A) = K(A) .$$

Indeed, if  $n \geq 0$ ,  $K(\pi A)_n = \coprod_{\theta: [n] \rightarrow [m]} A_m^{\theta}$  and it is enough to check that

$$D(\pi A)_n = \coprod_{\theta: [n] \rightarrow [m], \theta \neq Id} A_m^{\theta} .$$

By definition, the restriction of  $s_i : (\pi A)_{n-1} \rightarrow (\pi A)_n$  to the component  $A_m^{\sigma}$  corresponding to  $\sigma : [n-1] \rightarrow [m]$  is just

$$s_i|_{A_m^{\sigma}} = Id : A_m^{\sigma} \rightarrow A_m^{\sigma \circ s_i} .$$

Then, we deduce that  $D(\pi A)_n \subseteq \coprod_{\theta: [n] \rightarrow [m], \theta \neq Id} A_m^\theta$ .

The other inclusion is also clear since if  $\theta : [n] \rightarrow [m]$  is a non-identity surjection, then there exists  $0 \leq i \leq n-1$  and  $\tilde{\theta} : [n-1] \rightarrow [m]$  such that  $\theta = \tilde{\theta} \circ s_i$ . To see this, just take  $i$  such that  $\theta(i) = \theta(i+1)$  and define  $\theta'$  in the natural way.

*Step 2:* It holds that  $\pi : \Delta_e^\circ Set \rightarrow \Delta^\circ Set$  preserves quasi-isomorphisms.

Indeed, if  $f : W \rightarrow W'$  is a quasi-isomorphism, this means that  $K(Lf)$  is so in  $Ch_+ Ab$ , where  $L : \Delta_e^\circ Set \rightarrow \Delta_e^\circ Ab$  (or  $L : \Delta^\circ Set \rightarrow \Delta^\circ Ab$ ) is defined just by taking free groups. Note that  $\pi_* L = L \circ \pi$ .

From the previous step it follows that  $K^N(L\pi f) = K^N(\pi Lf) = K(Lf)$  is also a quasi-isomorphism. Since  $K^N$  and  $K$  are homotopic functors ([May] 22.3), we deduce that  $\pi f$  is a quasi-isomorphism as well.

*Step 3:* Given  $X \in \Delta^\circ Set$ , the morphism  $a_X : X \rightarrow \pi UX$  coming from the adjunction  $(\pi, U)$ , is a quasi-isomorphism.

The morphism  $a_X$  is in degree  $n$  the inclusion  $X_n \rightarrow X_n^{Id} \sqcup \coprod_{\theta: [n] \rightarrow [m], \theta \neq Id} X_m^\theta$ . Then by step 2 we get that  $K^N(La_X) : K^N(LX) \rightarrow K^N(\pi LX) = K(LX)$  coincides with the inclusion  $K^N(LX)_n \rightarrow K(LX)_n = K^N(LX)_n \oplus D(LX)_n$ , that again by loc. cit. is a homotopy equivalence.

*Step 4:* The functor  $U : Ho\Delta^\circ Set \rightarrow Ho\Delta_e^\circ Set$  is faithful.

Let  $f, g : X \rightarrow Y$  be morphisms in  $Ho\Delta^\circ Set$  such that  $Uf = Ug$  in  $Ho\Delta_e^\circ Set$ .

By step 2,  $\pi$  pass to the localized categories,  $\pi : Ho\Delta_e^\circ Set \rightarrow Ho\Delta^\circ Set$ .

Then  $\pi Uf = \pi Ug$  in  $Ho\Delta^\circ Set$ . On the other hand, it follows from the functoriality of  $a$  that  $\pi Uf \circ a_X = a_Y \circ f$ , and  $\pi Ug \circ a_X = a_Y \circ g$ . From step 3 we deduce that  $a_Y$  is an isomorphism in  $Ho\Delta^\circ Set$ , so  $f = g$  in  $Ho\Delta^\circ Set$ .  $\square$

To finish this section we give the following consequence of the properties previously developed of functors  $|\cdot|$  and  $\mathbf{s}$ .

**COROLLARY 5.5.13.** *The geometric realization  $|\cdot| : \Delta^\circ Set \rightarrow Top$  and the fat geometric realization  $\mathbf{s} : \Delta^\circ Set \rightarrow Top$  are functors of simplicial descent categories.*

*Proof.* Firstly, let us begin with  $\mathbf{s} : \Delta^\circ Set \rightarrow Top$ . The Eilenberg-Zilber theorem 5.5.10 provides the quasi-isomorphism  $\Theta = \mu_W : \mathbf{s}DW \rightarrow \mathbf{s}\Delta^\circ \mathbf{s}W$  for any bisimplicial set  $W$ . The compatibility between  $\Theta = \mu$  and the transformations  $\lambda$  of  $\Delta^\circ Set$  and  $Top$  follows from the compatibility between those transformations  $\lambda$  and  $\mu$  of  $Top$ .

On the other hand, the compatibility between  $\Theta$  and the respective transformations  $\mu$  can be deduced from the associativity property of the transformation  $\mu$  of  $Top$ .

To see that  $|\cdot| : \Delta^\circ Set \rightarrow Top$  is a functor of simplicial descent categories, we can take this time  $\Theta$  as the zig-zag whose value at a bisimplicial set  $W$  is

$$|DW| \xrightarrow{\eta_W} |\Delta^\circ W| \xleftarrow{\tau_{\Delta^\circ W}} s(\Delta^\circ W) ,$$

where  $\eta_W$  is the homeomorphism given in 5.5.10 and  $\tau_{\Delta^\circ W}$  is the homotopy equivalence from 5.5.9 (indeed  $\Delta^\circ W$  is a good simplicial topological space since it is obtained from bisimplicial set).

Consider now a simplicial set  $T$ . The compatibility between  $\Theta_T$  and the respective transformations  $\lambda_T$  is just the commutativity of the diagram

$$\begin{array}{ccccc} |D(\Delta \times T)| & \xrightarrow{\eta_{\Delta \times T}} & ||T| \times \Delta| & \xleftarrow{\tau_{|T| \times \Delta}} & s(|T| \times \Delta) \\ & \searrow Id & \downarrow \bar{\lambda}_T & \swarrow \lambda_{|T|} & \\ & & |T| & & \end{array}$$

where, if  $(t, p, q) \in T_n \times \Delta^n \times \Delta^m$  then  $\bar{\lambda}_T(|t, p, q|) = |t, p|$ .

On the other hand, given  $Z \in \Delta^\circ \Delta^\circ(\Delta^\circ Set)$ , under the notations of definition 2.5.10, we must check the commutativity in  $HoTop$  of the diagram

$$\begin{array}{ccccccc} sD_{1,2}\Delta^\circ\Delta^\circ|Z| & \xrightarrow{\tau_{\Delta^\circ|D_{1,2}Z|}} & |\Delta^\circ|D_{1,2}Z|| & \xleftarrow{\eta_{D_{1,2}Z}} & |DD_{1,2}Z| \\ \downarrow \mu_{\Delta^\circ\Delta^\circ|Z|} & & & & \downarrow Id \\ s\Delta^\circ s\Delta^\circ\Delta^\circ|Z| & \xrightarrow{s\Delta^\circ\tau} & s\Delta^\circ|\Delta^\circ\Delta^\circ|Z|| & \xleftarrow{s\Delta^\circ\eta} & s\Delta^\circ|D_{2,3}Z| & \xrightarrow{\tau} & |\Delta^\circ|D_{2,3}Z|| & \xleftarrow{\eta} & |DD_{2,3}Z| \end{array}$$

By the commutativity of the square

$$\begin{array}{ccc} s\Delta^\circ|D_{2,3}Z| & \xrightarrow{s\Delta^\circ\eta} & s\Delta^\circ|\Delta^\circ\Delta^\circ|Z|| \\ \tau \downarrow & & \downarrow \tau \\ |\Delta^\circ|D_{2,3}Z|| & \xrightarrow{|\Delta^\circ\eta|} & |\Delta^\circ|\Delta^\circ\Delta^\circ|Z|| \end{array} ,$$

we can just see the commutativity of

$$\begin{array}{ccccccc} sD_{1,2}\Delta^\circ\Delta^\circ|Z| & \xrightarrow{\tau_{\Delta^\circ|D_{1,2}Z|}} & |\Delta^\circ|D_{1,2}Z|| & \xleftarrow{\eta_{D_{1,2}Z}} & |DD_{1,2}Z| \\ \downarrow \mu_{\Delta^\circ\Delta^\circ|Z|} & & & & \downarrow Id \\ s\Delta^\circ s\Delta^\circ\Delta^\circ|Z| & \xrightarrow{\tau \circ s\Delta^\circ\tau} & |\Delta^\circ|\Delta^\circ\Delta^\circ|Z|| & \xleftarrow{|\Delta^\circ\eta| \circ \eta} & |DD_{2,3}Z| \end{array} ,$$



One can see this commutativity in  $HoTop$  dividing it into two squares through the map  $|\Delta^\circ\eta| : |\Delta^\circ|D_{1,2}Z|| \longrightarrow |\Delta^\circ|\Delta^\circ\Delta^\circ|Z||$ , and it follows from the definitions that these two squares commute in  $Top$ .  $\square$

## Chapter 6

# Examples of Cosimplicial Descent Categories

### 6.1 Cochain complexes

If  $\mathcal{A}$  is an additive category, the category  $Ch^*\mathcal{A}$  of cochain complexes can be identified with  $(Ch_*(\mathcal{A}^\circ))^\circ$ .

Assume moreover that  $\mathcal{A}$  has numerable products, that is, given a family  $\{A_k\}_{k \in \mathbb{Z}}$  of objects of  $\mathcal{A}$ , then  $\prod_{k \in \mathbb{Z}} A_k$  exists in  $\mathcal{A}$ .

In this case  $\mathcal{A}^\circ$  is an additive category with numerable products, so  $Ch_*(\mathcal{A}^\circ)$  is a simplicial descent category with respect to the homotopy equivalences.

We can argue analogously if  $\mathcal{A}$  is abelian with respect to the quasi-isomorphisms.

Again, we can drop the condition of existence of numerable products in the case of uniformly bounded-below cochain complexes.

Next we introduce the dual constructions to those given in 5.1.8.

#### DEFINITION 6.1.1.

**Simple functor:** The simple functor  $s : \Delta Ch^*(\mathcal{A}) \longrightarrow Ch^*(\mathcal{A})$  is defined as the composition

$$\Delta Ch^*(\mathcal{A}) \xrightarrow{K} Ch^*Ch^*(\mathcal{A}) \xrightarrow{Tot} Ch^*(\mathcal{A})$$

where  $K(\{X, d^i, s^j\}) = \{X, \sum (-1)^i d^i\}$ .

More concretely, let  $X = \{X^n, d^i, s^j\}$  be a cosimplicial cochain complex. Each  $X^n$  is an object of  $Ch^*(\mathcal{A})$ , that will be written as  $\{X^{n,p}, d^{X^n}\}_{p \in \mathbb{Z}}$ .

Then  $X$  induces the double cochain complex (6.1), with vertical boundary map  $d^{X^n} : X^{n,p} \rightarrow X^{n,p+1}$  and horizontal boundary map  $\partial : X^{n,p} \rightarrow X^{n+1,p}$ ,

$$\partial = \sum_{i=0}^{n+1} (-1)^i d^i.$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \dots & \longrightarrow & X^{n-1,p+1} & \xrightarrow{\partial} & X^{n,p+1} & \xrightarrow{\partial} & X^{n+1,p+1} \longrightarrow \dots \\
 & & \uparrow d^{X^{n-1}} & & \uparrow d^{X^n} & & \uparrow d^{X^{n+1}} \\
 \dots & \longrightarrow & X^{n-1,p} & \xrightarrow{\partial} & X^{n,p} & \xrightarrow{\partial} & X^{n+1,p} \longrightarrow \dots \\
 & & \uparrow d^{X^{n-1}} & & \uparrow d^{X^n} & & \uparrow d^{X^{n+1}} \\
 \dots & \longrightarrow & X^{n-1,p-1} & \xrightarrow{\partial} & X^{n,p-1} & \xrightarrow{\partial} & X^{n+1,p-1} \longrightarrow \dots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array} \tag{6.1}$$

Hence, the image of  $X$  under the simple functor is the cochain complex  $\mathbf{s}X$  given by

$$(\mathbf{s}X)^q = \prod_{p+n=q} X^{n,p} \quad d = \prod (-1)^p \partial + d^{X^n} : \prod_{p+n=q} X^{n,p} \longrightarrow \prod_{p+n=q+1} X^{n,p}.$$

**Transformation  $\lambda$ :** If  $A \in Ch^*(\mathcal{A})$ ,  $\mathbf{s}(A \times \Delta)$  is in degree  $n$  the product  $\prod_{k \leq n} A^k$ , in such a way that the inclusion  $\lambda_A : A \rightarrow \mathbf{s}(A \times \Delta)$  is a morphism of cochain complexes.

**Transformation  $\mu$ :** If  $Z \in \Delta\Delta Ch^*(\mathcal{A})$ ,  $\mu_Z : \mathbf{s}\Delta^\circ \mathbf{s}(Z) \rightarrow \mathbf{s}D(Z)$  is in degree  $n$

$$(\mu_Z)^n = \prod (\mu_Z)^{p,q} : \prod_{i+j+q=n} Z^{i,j,q} \rightarrow \prod_{p+q=n} Z^{p,p,q}$$

where, given  $p, q$  with  $p + q = n$ ,  $(\mu_Z)^{p,q}$  is

$$(\mu_Z)^{p,q} = \sum_{i+j=p} Z(d^0 \overset{j}{\dots} d^0, d^p d^{p-1} \dots d^{j+1}) : Z^{i,j,q} \rightarrow Z^{p,p,q}.$$

If  $p$  is a fixed integer, note that the sum  $\sum_{i+j=p} Z(d^0 \overset{j}{\dots} d^0, d^p d^{p-1} \dots d^{j+1})$  is finite since the indexes  $i, j$  in  $Z^{i,j,q}$  are positive (because  $Z \in \Delta\Delta Ch^*(\mathcal{A})$ ).

Thus, the next proposition follows directly from the definition of cosimplicial descent category.

**PROPOSITION 6.1.2.** *Under the above notations, if  $\mathcal{A}$  is an additive category with numerable products, then  $(Ch^* \mathcal{A}, \mathbf{s}, \mu, \lambda)$  is a cosimplicial descent category with respect to the homotopy equivalences.*

If moreover  $\mathcal{A}$  is abelian then  $(Ch^*\mathcal{A}, s, \mu, \lambda)$  is a cosimplicial descent category, where  $E$  is the class of quasi-isomorphisms. In both descent structures  $\lambda$  is quasi-invertible and  $\mu$  is associative.

This time we will give the specific consequences of the previous proposition, because while the cone in  $Ch^*\mathcal{A}$  is widely known, does not occur the same to its dual construction, the path object.

**PROPOSITION 6.1.3.** *Given morphisms  $A \xrightarrow{f} B \xleftarrow{g} C$  of cochain complexes, the path object associated with  $f$  and  $g$  is a cochain complex  $path(f, g)$ , functorial in the pair  $(f, g)$ , which satisfies the following properties*

1) *there exists maps in  $Ch^*\mathcal{A}$ , functorial in  $(f, g)$*

$$j_A : path(f, g) \rightarrow A \quad j_B : path(f, g) \rightarrow B$$

*such that  $j_A$  is a quasi-isomorphism (resp. homotopy equivalence) if and only if  $g$  is so. Similarly,  $j_B$  is a quasi-isomorphism (resp. homotopy equivalence) if and only if  $f$  is so.*

2) *If  $f = g = Id_A$ , there exists a homotopy equivalence  $P : A \rightarrow path(A)$  in  $Ch^*\mathcal{A}$  such that the composition of  $P$  with the projections  $j_A, j'_A : path(A) \rightarrow A$  given in 1) is equal to the identity.*

3) *The following square commutes up to homotopy equivalence*

$$\begin{array}{ccc} B & \xleftarrow{f} & A \\ \uparrow g & & \uparrow j_A \\ C & \xleftarrow{j_C} & path(f, g) \end{array} .$$

**REMARK 6.1.4.** When  $C = 0$ ,  $Path(f, 0)$  is (up to homotopy equivalence) the cochain complex  $c(f)[-1]$ , where  $c(f)$  is the classical cone of  $f$ .

The proof is just the dual of the one where we checked the commutativity of diagram (5.4) in proposition 5.1.9 up to homotopy, and can be found in [H] 2.2.11.

Then, under the classical approach of the homotopy theory of  $Ch^*\mathcal{A}$ ,  $f$  gives rise to the distinguished canonical triangle

$$A \xrightarrow{f} B \xrightarrow{i} c(f) \xrightarrow{p} A[1]$$

and, under our settings, the morphism  $j_A : Path(f, 0) \rightarrow A$  corresponds to the projection  $c(f)[-1] \xrightarrow{p[-1]} A$ . Therefore the induced triangulated structure on  $HoCh^*\mathcal{A}$  coincides with the usual one.

**LEMMA 6.1.5.** *Given two morphisms  $A \xrightarrow{f} B \xleftarrow{g} C$  in  $Ch^*(\mathcal{A})$ , there exists a natural homotopy equivalence between  $path(f, g)$  and the cochain complex  $path_r(f, g)$  given by*

$$path_r(f, g)^n = A^n \oplus B^{n-1} \oplus C^n ; d = \begin{pmatrix} d_A & 0 & 0 \\ f & -d_B & g \\ 0 & 0 & 0 \end{pmatrix} .$$

*In addition, the morphisms  $j_A$  and  $j_C$  correspond to the respective projections  $A^n \oplus B^{n-1} \oplus C^n \rightarrow A^n$  and  $A^n \oplus B^{n-1} \oplus C^n \rightarrow C^n$ .*

*Idea of the proof.* By definition  $path(f, g) = s(Path(f \times \Delta, g \times \Delta))$ . The cosimplicial object  $Path(f \times \Delta, g \times \Delta)$  is the image under the total functor (dual of definition 1.3.6) of  $T$ , consisting of the biaugmented bisimplicial cochain complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ A & \xrightarrow{f} & B & \rightleftarrows & B & \rightleftarrows & B \\ \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow \\ A & \xrightarrow{f} & B & \rightleftarrows & B & \rightleftarrows & B \\ \uparrow \uparrow & & \uparrow \uparrow & & \uparrow \uparrow & & \uparrow \uparrow \\ A & \xrightarrow{f} & B & \rightleftarrows & B & \rightleftarrows & B \\ & & \uparrow g & & \uparrow g & & \uparrow g \\ & & C & \rightleftarrows & C & \rightleftarrows & C \end{array} \quad \dots$$

where all maps without label are identities. Then, it follows from the definitions that  $K(Tot(T))$  coincides with  $Tot \circ K(T)$ . Therefore  $path(f, g) = Tot(Tot(K(T)))$  is the total cochain complex associated with the following

triple complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
A & \xrightarrow{f} & B & \xrightarrow{0} & B & \xrightarrow{Id} & B & \xrightarrow{0} & B & \cdots \\
\uparrow Id & & \uparrow Id & & \uparrow Id & & \uparrow Id & & \uparrow Id & \\
A & \xrightarrow{f} & B & \xrightarrow{0} & B & \xrightarrow{Id} & B & \xrightarrow{0} & B & \cdots \\
\uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
A & \xrightarrow{f} & B & \xrightarrow{0} & B & \xrightarrow{Id} & B & \xrightarrow{0} & B & \cdots \\
& & \uparrow g & & \uparrow g & & \uparrow g & & \uparrow g & \\
& & C & \xrightarrow{0} & C & \xrightarrow{Id} & C & \xrightarrow{0} & C & \cdots
\end{array}$$

which is homotopic by columns to

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A & \xrightarrow{f} & B & \xrightarrow{0} & B & \xrightarrow{Id} & B & \xrightarrow{0} & B & \xrightarrow{0} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C & \xrightarrow{0} & C & \xrightarrow{Id} & C & \xrightarrow{0} & C & \xrightarrow{0} & C & \xrightarrow{0} & \cdots
\end{array}$$

that is homotopic by rows to

(6.2)

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
A & \xrightarrow{f} & B & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
& & \uparrow g & & \uparrow & & \uparrow & \\
& & C & \longrightarrow & 0 & \longrightarrow & 0 & \cdots
\end{array}$$

Therefore, the projection of  $path(f, g)$  onto the total complex of (6.2) is a homotopy equivalence. But this total complex coincides with  $path_r(f, g)$ , so  $j_A$  and  $j_C$  correspond to the projections of  $path_r(f, g)$  onto  $A$  and  $C$ , respectively.

☐

*Proof of 6.1.3.* The statement follows from proposition 3.2.2. Let us see 3). By loc. cit. there exists  $H : \text{path}(f, g) \rightarrow \text{path}(B)$  such that  $j'_B \circ H = f \circ j_A$  and  $j''_B \circ H = g \circ j_B$ , where  $j'_B, j''_B : \text{path}(A) \rightarrow A$  are the canonical projections. Then, it suffices to see that  $j'_B$  and  $j''_B$  are homotopic morphisms of  $\text{Ch}^* \mathcal{A}$ , or equivalently, that the projections  $P, Q : \text{path}_r(\text{Id}_B, \text{Id}_B) \rightarrow B$  are homotopic. To this end, consider the homotopy  $H : \text{path}_r(\text{Id}_B, \text{Id}_B)^n \rightarrow B^{n-1}$  defined as the projection onto the second summand  $B^n \oplus B^{n-1} \oplus B^n \rightarrow B^{n-1}$ .  $\square$

**REMARK 6.1.6.** Similarly to the chain complexes case, in the abelian we can consider as well the normalized version of the simple functor,  $\mathbf{s}_N$ .

The functor  $\text{path}$  obtained using the normalized simple functor is equal to  $\text{path}_r$ .

It also holds the dual of propositions 5.2.5 and corollary 5.2.7.

**PROPOSITION 6.1.7.** *Let  $\mathcal{A}$  be an additive category (resp. abelian). Then the category  $\text{Ch}^q \mathcal{A}$  of uniformly bounded-below cochain complexes, together with the homotopy equivalences (resp. the quasi-isomorphisms) as equivalences and the data  $\lambda, \mu$  given in 6.1.1, is an additive cosimplicial descent category. In addition,  $\lambda$  is quasi-invertible and  $\mu$  is associative.*

We obtain in this way the usual “cosuspended” (or left triangulated) category structure on  $\text{HoCh}^q \mathcal{A}$ .

**COROLLARY 6.1.8.** *Let  $\text{Ch}^b \mathcal{A}$  be the category of bounded-below cochain complexes. Then the localized category  $\text{HoCh}^b \mathcal{A}$  of  $\text{Ch}^b \mathcal{A}$  with respect to the quasi-isomorphisms (resp. homotopy equivalences) is a triangulated category.*

## 6.2 Commutative differential graded algebras

The Thom-Whitney functor and its properties were developed in [N]. This simple functor gives rise to a cosimplicial descent category structure on the category of commutative differential graded algebras over a field of characteristic 0.

### DEFINITION 6.2.1.

Let  $\mathbf{Cdga}(k)$  be the category of commutative differential graded algebras (or cdg algebras) over a field  $k$  of characteristic 0.

The product of two cdg algebras  $A$  and  $B$  has as underlying cochain complex the product (i.e. the direct sum) of those underlying complexes of  $A$  and  $B$ .

### DEFINITION 6.2.2 (Descent structure on $\mathbf{Cdga}(k)$ ).

**Simple functor:** The Thom-Whitney simple functor  $\mathbf{s}_{TW} : \Delta \mathbf{Cdga}(k) \rightarrow$

$\mathbf{Cdga}(k)$ , introduced in [N], is defined as follows.

Firstly, let  $L \in \Delta^\circ \mathbf{Cdga}(k)$  [BG] be the simplicial object of  $\mathbf{Cdga}(k)$  such that

$$L_n = \frac{\Lambda(x_0, \dots, x_n, dx_0, \dots, dx_n)}{(\sum x_i - 1, \sum dx_i)},$$

where  $\Lambda_n = \Lambda(x_0, \dots, x_n, dx_0, \dots, dx_n)$  is the free cdg algebra in which  $x_k$  has degree 0 and  $dx_k$  has degree 1,  $0 \leq k \leq n$ . The boundary map is the unique derivation in  $\Lambda_n$  such that  $d(x_k) = dx_k$ ,  $d(dx_k) = 0$ .

The face maps  $d_i : L_{n+1} \rightarrow L_n$  and the degeneracy maps  $s_j : L_n \rightarrow L_{n+1}$  are

$$\text{defined as } d_i(x_k) = \begin{cases} x_k, & k < i \\ 0, & k = i \\ x_{k-1}, & k > i \end{cases} \text{ and } s_j(x_k) = \begin{cases} x_k, & k < j \\ x_k + x_{k+1}, & k = j \\ x_{k+1}, & k > j \end{cases}.$$

Given  $A \in \Delta \mathbf{Cdga}(k)$ , denote by  $T_A : \Delta_e^\circ \times \Delta_e \rightarrow \mathbf{Cdga}(k)$  the bifunctor obtained from  $L \otimes A : \Delta^\circ \times \Delta \rightarrow \mathbf{Cdga}(k)$  by forgetting the degeneracy maps. Then the Thom-Whitney simple is the final

$$\mathbf{s}_{TW}(A) = \int_n T_A(n, n).$$

**Equivalences:** The class E consists of the quasi-isomorphisms, that is to say, those morphisms of cdg algebras which induce isomorphism in cohomology.

**Transformation  $\lambda$ :** If  $A \in \mathbf{Cdga}(k)$ , the morphisms  $A \rightarrow A \otimes L_n$ ;  $a \rightarrow a \otimes 1$  define the morphism  $\lambda(A) : A \rightarrow \mathbf{s}_{TW}(A \times \Delta)$ .

**Transformation  $\mu$ :** If  $Z \in \Delta \Delta \mathbf{Cdga}(k)$ ,  $\mu_{TW}(Z) : \mathbf{s}_{TW} \Delta^\circ \mathbf{s}_{TW} Z \rightarrow \mathbf{s}_{TW} DZ = \int_p Z^{p,p} \otimes L_p$  is given by the morphisms

$$\mathbf{s}_{TW} \Delta^\circ \mathbf{s}_{TW} Z \xrightarrow{\pi} Z^{p,p} \otimes L_p \otimes L_p \xrightarrow{Id \otimes \tau_p} Z^{p,p} \otimes L_p$$

where  $\pi$  is the iterated projection and  $\tau_p : L_p \otimes L_p \rightarrow L_p$  is the structural morphisms of the cdg algebra  $L_p$ , that are morphisms of cdg algebras since  $L_p$  is commutative.

**PROPOSITION 6.2.3.** *The category  $\mathbf{Cdga}(k)$  together with the quasi-isomorphisms and the Thom-Whitney simple is a cosimplicial descent category. In addition, the forgetful functor  $U : \mathbf{Cdga}(k) \rightarrow Ch^*k$  is a functor of cosimplicial descent categories.*

We will use the following notations in the proof of the previous proposition.

**(6.2.4)** Let  $k[p]$  be the cochain complex that is equal to 0 in all degrees except  $p$ , and  $(k[p])^p = k$ . As in [N] (2.2), we denote by  $\int_{\Delta^p} : L_p \rightarrow k[p]$  the map



of cochain complexes which in degree  $p$  is the “integral over the simplex  $\Delta^p$ ”,  $L_p^p \rightarrow k$ .

*Proof.* Let  $Ch^*k$  be the category of cochain complexes of  $k$ -vector spaces. Let us check that the forgetful functor  $U : \mathbf{Cdga}(k) \rightarrow Ch^*k$  satisfies the dual proposition to 2.5.8, where  $Ch^*k$  is considered with the usual descent structure given in definition 6.1.1.

The axiom  $(FD1)^\circ$  hold, and it is straightforward to check that  $\lambda$  and  $\mu$  are compatible natural transformations.

Let us see  $(FD2)^\circ$ . Denote by  $s_{TW} : \Delta Ch^*k \rightarrow Ch^*k$  the functor defined as the final of  $U(L) \otimes A : \Delta_e^\circ \times \Delta_e \rightarrow Ch^*k$ . Then we have the commutative diagram

$$\begin{array}{ccc} \Delta \mathbf{Cdga}(k) & \xrightarrow{\Delta U} & \Delta Ch^*k \\ \mathbf{s}_{TW} \downarrow & & \downarrow \mathbf{s}_{TW} \\ \mathbf{Cdga}(k) & \xrightarrow{U} & Ch^*k. \end{array}$$

By ([N], 2.15), there exists a natural transformation  $I : \mathbf{s}_{TW} \rightarrow \mathbf{s} : \Delta Ch^*k \rightarrow Ch^*k$  such that  $I(A)$  is a homotopy equivalence for each  $A \in Ch^*k$ . Set  $\Theta = I \Delta U : U \mathbf{s}_{TW} \rightarrow \mathbf{s} \Delta U : \Delta \mathbf{Cdga}(k) \rightarrow Ch^*k$ . More concretely, given  $A \in \Delta \mathbf{Cdga}(k)$ , the morphism  $\Theta_A : U \mathbf{s}_{TW}(A) \rightarrow \mathbf{s} \Delta U(A)$  in degree  $n$  is the map  $(\mathbf{s}_{TW}(A))^n \rightarrow (\mathbf{s} \Delta U(A))^n = \prod_{p+q=n} A^{p,q}$  whose projection onto the component  $A^{p,q}$  is given by the composition

$$(\mathbf{s}_{TW}(A))^n = \int_m (A^m \otimes L_m)^n \xrightarrow{\pi} (A^p \otimes L_p)^n \xrightarrow{Id \otimes \int_{\Delta^p}} A^{p,q},$$

where  $\pi$  denotes the projection and  $\int_{\Delta^p}$  is the morphism introduced in 6.2.4.

The compatibility between  $\lambda : Id_{\mathbf{Cdga}(k)} \rightarrow \mathbf{s}_{TW}(- \times \Delta)$  and  $\lambda' : Id_{Ch^*k} \rightarrow \mathbf{s}(- \times \Delta)$  means the commutativity up to quasi-isomorphism of the diagram

$$\begin{array}{ccc} U(\mathbf{s}_{TW}(A \times \Delta)) & \xleftarrow{U(\lambda(A))} & U(A). \\ \Theta(A \times \Delta) \downarrow & \swarrow \lambda'(U(X)) & \\ \mathbf{s}(U(A) \times \Delta) & & \end{array}$$

that in degree  $n$  becomes

$$\begin{array}{ccc}
 \mathbf{s}_{TW}^n(A \times \Delta) & \xleftarrow{\lambda_n(A)} & A^n. \\
 \Theta_n(A \times \Delta) \downarrow & \nearrow i_n & \\
 \prod_{i \leq n} A^i & & 
 \end{array}$$

If  $a \in A^n$ ,  $\lambda_n(A)(a) = \{(1_m \otimes a, 0, \dots)\}_{m \geq 0} \in \mathbf{s}_{TW}^n(A \times \Delta) \subset \prod_{k+l=n} \prod_m L_m^k \otimes A^l$ , where  $1_m \in L_m^0$  is the unit of  $L_m$ . Therefore

$$\Theta_n(A \times \Delta)(\lambda_n(A)(a)) = ((\int_{\Delta^0} 1)a, 0, \dots) = (a, 0, \dots) = i_n(a).$$

It remains to see that  $\mu_{TW} : \mathbf{s}_{TW} \Delta \mathbf{s}_{TW} \rightarrow \mathbf{s}_{TW} D$ ,  $\mu' : \mathbf{s} \Delta \mathbf{s} \rightarrow \mathbf{s} D$  and  $\Theta$  are compatible. To see this, it is enough to prove the commutativity up to homotopy of the diagram

$$\begin{array}{ccc}
 \mathbf{U} \mathbf{s}_{TW} \Delta \mathbf{s}_{TW}(A) & \xrightarrow{\mathbf{U}(\mu_{TW})} & \mathbf{U} \mathbf{s}_{TW}(DA) \\
 \tilde{\Theta} \downarrow & & \downarrow \Theta_{DA} \\
 \mathbf{s}(\Delta \mathbf{s} \Delta \Delta \mathbf{U}(A)) & \xrightarrow{\mu'} & \mathbf{s} D(\Delta \Delta \mathbf{U}(A)),
 \end{array} \tag{6.3}$$

where  $\tilde{\Theta}$  is the composition

$$\mathbf{U} \mathbf{s}_{TW} \Delta \mathbf{s}_{TW}(A) \xrightarrow{\Theta_{\Delta \mathbf{s}_{TW}(A)}} \mathbf{s}(\Delta \mathbf{U} \mathbf{s}_{TW} A) \xrightarrow{\mathbf{s}(\Delta \Theta_A)} \mathbf{s}(\Delta \mathbf{s} \Delta \Delta \mathbf{U}(A)).$$

Using the homotopy inverse  $E$  of  $\Theta$  given in [N], one can obtain a homotopy inverse of  $\tilde{\Theta}$ , that will be referred to as  $\tilde{E}$ . Then, it holds that  $\Theta_{DA} \circ \mathbf{U}(\mu_{TW}) \circ \tilde{E}$  coincides with  $\mu'$  up to homotopy equivalence.

Indeed, (6.3) is homotopy equivalent to the image under the total complex of a square of double cochain complexes of  $k$ -vector spaces of the form

$$\begin{array}{ccc}
 \mathbf{U} \mathbf{s}_{TW}^{**}(A) & \xrightarrow{\mathbf{U}(\mu_{TW}^{**})} & \mathbf{U} \mathbf{s}_{TW}^{**}(DA) \\
 E^{**} \uparrow & & \downarrow I^{**} \\
 \mathbf{s}^{**} \mathbf{U}(A) & \xrightarrow{\mu_{E-Z}} & K D \mathbf{U}(A).
 \end{array}$$

where  $\mu_{E-Z}$  is the Alexander-Whitney map (see A.1.3). By the Eilenberg-Zilber-Cartier theorem A.1.1, it is enough to check that the above square commutes in degree 0, but this is a straightforward calculation, totally similar to [N], (3.4).  $\square$

### 6.2.1 Comments on the non-commutative case

Consider now a commutative (associative and unitary) ring  $R$ , and let  $\mathbf{Dga}$  be the category of differential graded  $R$ -algebras (not necessarily commutative). Denote by  $Ch^*R$  the category of cochain complexes of  $R$ -modules.

In this case, as we will explain later, the simple functor in  $Ch^*R$  (definition 6.1.1) induces in a natural way the so called Alexander-Whitney simple  $\mathbf{s}_{AW} : \Delta\mathbf{Dga} \rightarrow \mathbf{Dga}$  [N].

Then, a natural question is if this Alexander-Whitney simple provides a descent structure on  $\mathbf{Dga}$ , together with the quasi-isomorphisms. The answer is that all axioms of cosimplicial descent category are satisfied by this simple, except the factorization one.

The reason is that the transformation  $\mu$  of the cosimplicial descent structure on  $Ch^*R$  does not induce a natural transformation in  $\mathbf{Dga}$ , because it is not compatible with the multiplicative structures involved.

Hence, the descent structure on  $Ch^*R$  does not induce a descent structure on  $\mathbf{Dga}$ . We will give in this subsection an explicit counterexample of this fact. That is, we will exhibit a bicosimplicial graded algebra  $Z$  such that the morphism of cochain complexes  $\mu_Z : \mathbf{s}\Delta\mathbf{s}Z \rightarrow \mathbf{s}D$  does not preserve the multiplicative structure, so it is not a morphism of the category  $\mathbf{Dga}$ .

(6.2.5) By definition 6.1.1, the simple functor  $\mathbf{s} : \Delta Ch^*R \rightarrow Ch^*R$  is the composition of functors  $K$  and  $Tot$ . Since both functors are monoidal with respect to the tensorial product of  $R$ -modules, so is  $\mathbf{s}$ . Hence, given cosimplicial cochain complexes  $X$  and  $Y$ , we have the Künneth morphism

$$k : \mathbf{s}X \otimes \mathbf{s}Y \longrightarrow \mathbf{s}(X \otimes Y)$$

that is obtained from the Alexander-Whitney map (see A.1.3) as follows. In degree  $n$ ,

$$k^n : \bigoplus_{p+q=n} \left( \prod_{i+j=p} X^{i,j} \otimes \prod_{s+t=q} Y^{s,t} \right) \longrightarrow \prod_{l+m=n} (X^l \otimes Y^l)^m$$

where, as usual,  $X^d \in Ch^*R$  is denoted by  $\{X^{d,r}\}_{r \in \mathbb{Z}}$  for any  $d \geq 0$ . Then  $k^n$  is determined by the morphisms

$$\sum_{u+v=l} k^{u,v} : \prod_{i+j=p} X^{i,j} \otimes \prod_{s+t=q} Y^{s,t} \longrightarrow (X^l \otimes Y^l)^m ,$$

If  $u, v \geq 0$  with  $u + v = l$ ,  $k^{u,v}$  is given by the composition

$$\prod_{i+j=p} X^{i,j} \otimes \prod_{s+t=q} Y^{s,t} \xrightarrow{p \otimes p} X^{u,p-u} \otimes Y^{v,q-v} \xrightarrow{A-W} X^{u+v,p-u} \otimes Y^{u+v,q-v}$$

where each  $p$  denotes the corresponding projection, whereas  $A - W = (-1)^{uq} X(d^0 \dots d^0) \otimes Y(d^l d^{l-1} \dots d^{v+1})$ . The sign  $(-1)^{uq}$  appearing in the last equation comes from the Künneth morphism of the functor  $Tot$ .

**DEFINITION 6.2.6** (Alexander-Whitney Simple). [N]

Let  $A \in \Delta \mathbf{Dga}$  and  $UA \in \Delta Ch^*R$  be the cosimplicial cochain complex obtained by forgetting the multiplicative structure.

We have that  $\mathbf{s}(UA)$  is a differential graded algebra through the morphism  $\tau_A : \mathbf{s}(UA) \otimes \mathbf{s}(UA) \rightarrow \mathbf{s}(UA)$  defined as the composition

$$\mathbf{s}(UA) \otimes \mathbf{s}(UA) \xrightarrow{k} \mathbf{s}(UA \otimes UA) \xrightarrow{s\tau} \mathbf{s}(A) \quad (6.4)$$

where  $k$  is the Künneth morphism and  $\tau^n : UA^n \otimes UA^n \rightarrow UA^n$  is the structural morphism of the differential graded algebra  $A^n$ .

The Alexander-Whitney simple ([N], 3.1) is the functor  $\mathbf{s}_{AW} : \Delta \mathbf{Dga} \rightarrow \mathbf{Dga}$  obtained in this way.

**REMARK 6.2.7.** Consider now  $B \in \mathbf{Dga}$  and  $Z \in \Delta \Delta \mathbf{Dga}$ . We have the following morphisms in  $Ch^*R$

$$\lambda_{UB} : UB \rightarrow \mathbf{s}(B \times \Delta) \quad \mu_{UZ} : \mathbf{s}DUZ \rightarrow \mathbf{s}\Delta \mathbf{s}UZ .$$

It can be checked easily that  $\lambda_{UB}$  is compatible with the respective multiplicative structures of  $B$  and  $\mathbf{s}_{AW}(B \times \Delta)$ , giving rise to a natural transformation  $\lambda_{AW} : Id_{\mathbf{Dga}} \rightarrow \mathbf{s}_{AW}(- \times \Delta)$ .

However, it does not happen the same with  $\mu$ , as we will see in the following counterexample.

**EXAMPLE 6.2.8.** Consider  $\mathbb{Z}$  as a differential graded algebra concentrated in degree 0 and let  $\mathbb{Z} \times \Delta \in \Delta \mathbf{Dga}$  be the associated constant cosimplicial object. Denote by  $B \in \Delta \mathbf{Dga}$  the path object associated with  $\mathbb{Z} \times \Delta$ . In other words,

$B$  is equal to  $Path(\mathbb{Z} \leftarrow \mathbb{Z} \times \Delta \rightarrow \mathbb{Z} \times \Delta)$  (definition 1.5.16), and can be visualized as

$$\mathbb{Z} \times \mathbb{Z} \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{d^1} \end{array} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \quad \dots$$

where  $d^0(a, b) = (a, a, b)$  and  $d^1(a, b) = (a, b, b)$ , for any integers  $a$  and  $b$ .

Set  $Z = B \times \Delta \in \Delta\Delta\mathbf{Dga}$ , that is,  $Z^{n,m} = B^n = \mathbb{Z} \times \overset{n+2}{\dots} \times \mathbb{Z}$ . In this case the morphism of cochain complexes  $\mu_Z : \mathbf{s}_{AW}DZ \rightarrow \mathbf{s}_{AW}\Delta\mathbf{s}_{AW}Z$  is not a morphism of algebras.

Indeed, if we consider  $x = (1, 0) \in Z^{0,1,0} \subset (\mathbf{s}_{AW}\Delta\mathbf{s}_{AW}Z)^1$  and  $y = (1, 2) \in Z^{0,0,0} \subset (\mathbf{s}_{AW}\Delta\mathbf{s}_{AW}Z)^0$ , it holds that  $\mu(x) \cdot \mu(y) \neq \mu(x \cdot y)$ , (where each  $\cdot$  denotes the corresponding product in  $\mathbf{s}_{AW}DZ$  and  $\mathbf{s}_{AW}\Delta\mathbf{s}_{AW}Z$ ).

Let us compute first  $\mu(x) \cdot \mu(y)$ .

By definition (see 6.1.1) we have that  $\mu(x) = Z(d^0, Id)x = (1, 1, 0) \in Z^{1,1,0} \subset (\mathbf{s}_{AW}DZ)^1$  and  $\mu(y) = Z(Id, d^1)y = y \in Z^{0,0,0} \subset (\mathbf{s}_{AW}DZ)^0$ . The product of these two elements is, following (6.4), equal to the product in  $Z$  of  $(1, 1, 0)$  and  $Z(d^1, d^1)y = (1, 2, 2)$ , so  $\mu(x) \cdot \mu(y) = (1, 1, 0) \cdot (1, 2, 2) = (1, 2, 0) \in Z^{1,1,0} \subset (\mathbf{s}_{AW}DZ)^1$ .

Secondly,  $x \cdot y$  is the product in  $Z$  of  $x \in Z^{0,1,0}$  and  $Z(Id, d^1)y = y \in Z^{0,1,0}$ , that is,  $x \cdot y = (1, 0) \in Z^{0,1,0}$ .

Therefore  $\mu(x \cdot y) = Z(d^0, Id)(1, 0) = (1, 1, 0) \in Z^{1,1,0}$ .

Consequently  $\mu(x) \cdot \mu(y) = (1, 2, 0) \neq (1, 1, 0) = \mu(x \cdot y)$ .

**REMARK 6.2.9.** Given  $Z \in \Delta\Delta\mathbf{Dga}$ , let  $T$  be the 4-simplicial object in  $\mathbf{Dga}$  given by  $T^{i,j,k,l} = Z^{i,j} \otimes Z^{k,l}$ . Under the notations of 2.5.10,  $Z \otimes Z = D^{1,3}D^{2,4}T$ . In addition  $\mathbf{s}\Delta\mathbf{s}Z \otimes \mathbf{s}\Delta\mathbf{s}Z$  is obtained by applying four times  $\mathbf{s}$  to  $T$ .

In this way, the Künneth morphism  $k : \mathbf{s}\Delta\mathbf{s}Z \otimes \mathbf{s}\Delta\mathbf{s}Z \rightarrow \mathbf{s}\Delta\mathbf{s}(Z \otimes Z)$  is just an iteration of  $\mu$ . Analogously, under this point of view,  $k : \mathbf{s}DZ \otimes \mathbf{s}DZ \rightarrow \mathbf{s}D(Z \otimes Z)$  is just the image under  $\mu$  of  $D^{1,2}D^{3,4}T$ .

The preservation of the multiplicative structure by  $\mu$  means the commutativity of the diagram

$$\begin{array}{ccccc} \mathbf{s}\Delta\mathbf{s}Z \otimes \mathbf{s}\Delta\mathbf{s}Z & \xrightarrow{k} & \mathbf{s}\Delta\mathbf{s}(Z \otimes Z) & \xrightarrow{\mathbf{s}\Delta\mathbf{s}\tau} & \mathbf{s}\Delta\mathbf{s}Z \\ \mu \otimes \mu \downarrow & & \mu_{Z \otimes Z} \downarrow & & \mu_Z \downarrow \\ \mathbf{s}DZ \otimes \mathbf{s}DZ & \xrightarrow{k} & \mathbf{s}D(Z \otimes Z) & \xrightarrow{\mathbf{s}D\tau} & \mathbf{s}DZ \end{array}$$

The right hand side commutes by the naturality of  $\mu$ , but the left hand side commutes provided that  $\mu$  is associative and commutative. This is not the case

because the Alexander-Whitney map fails to be commutative, then  $\mu$  is not a morphism of algebras.

In the commutative case we have used the Thom-Whitney simple. The corresponding natural transformation  $\mu$  comes from the product in the cdg algebra  $L$ , so this time  $\mu$  is actually an associative and commutative natural transformation.

### 6.3 DG-modules over a DG-category

In this section we study the category of DG-modules over a fixed DG-category  $[K]$  as an example of cosimplicial descent category. We begin by recalling the definition of DG-category.

Along this section  $R$  will be a fixed commutative ring, and the tensorial product  $\otimes_R$  over  $R$  will be written as  $\otimes$ .

**DEFINITION 6.3.1.** If  $\mathcal{A}$  is a category and  $A, B$  are objects of  $\mathcal{A}$ , denote by  $\mathcal{A}(A, B)$  the set of morphisms of from  $A$  to  $B$  in the category  $\mathcal{A}$ .

A DG-category  $\mathcal{A}$  (or a *differential graded category*) is a category such that given objects  $A, B$  of  $\mathcal{A}$  then

$$\mathcal{A}(A, B) = \{\mathcal{A}(A, B)^r\}_{r \in \mathbb{Z}}$$

where each  $\mathcal{A}(A, B)^r$  is an  $R$ -module.

Moreover,  $\mathcal{A}(A, B)$  has a boundary map  $d : \mathcal{A}(A, B)^r \rightarrow \mathcal{A}(A, B)^{r+1}$  satisfying the following properties

- 0. the composition of morphisms of  $\mathcal{A}$  is a homogeneous map of degree 0

$$\mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \rightarrow \mathcal{A}(A, C)$$

- 1.  $d^2 = 0$ , that is,  $\mathcal{A}(A, B)$  is a cochain complex of  $R$ -modules.

- 2. if  $f$  and  $g$  are composable morphisms of  $\mathcal{A}$  and  $f$  is homogeneous of degree  $p$  then

$$d(f \circ g) = (df) \circ g + (-1)^p f \circ (dg) .$$

**EXAMPLE 6.3.2.** The DG-category  $\text{Dif } R$  has as objects the cochain complexes of  $R$ -modules. If  $V$  and  $W$  are such cochain complexes, set  $\text{Dif } R(V, W) = \{\text{Dif } R(V, W)^p\}_{p \in \mathbb{Z}}$  where

$$\text{Dif } R(V, W)^p = \{f : V \rightarrow W \text{ morphism of } R\text{-modules} \mid f(V^k) \subseteq W^{p+k}\} .$$

Denote by  $f^k = f|_{V^k} : V^k \rightarrow W^{p+k}$ .

The image under the boundary map  $d : \text{Dif } R(V, W)^p \rightarrow \text{Dif } R(V, W)^{p+1}$  of  $f = \{f^k\}_{k \in \mathbb{Z}}$  is

$$\{d^W \circ f^k - (-1)^p f^{k+1} \circ d^V\}_{k \in \mathbb{Z}}.$$

**REMARK 6.3.3.** Note that a morphism  $f \in \text{Dif } R(V, W)^p$  does not commute in general with the boundary maps of  $V$  and  $W$ . Actually, the cochain complex

$$\cdots \longrightarrow \text{Dif } R(V, W)^{-1} \xrightarrow{d^{-1}} \text{Dif } R(V, W)^0 \xrightarrow{d^0} \text{Dif } R(V, W)^1 \longrightarrow \cdots$$

is such that  $\text{Ker } d^1$  consists of the morphisms of cochain complexes between  $V$  and  $W$ , whereas  $H^0(\text{Dif } R(V, W))$  consists of the morphisms between them in the homotopy category  $K(R\text{-mod})$ , that is, are equivalence classes of morphisms of complexes modulo homotopy.

From now until the end of this section  $\mathcal{A}$  will denote a fixed DG-category, that we will assume to be a small category.

**DEFINITION 6.3.4.** The category  $\mathcal{CA}$  of *differential graded modules* over  $\mathcal{A}$  has as objects the functors of DG-categories

$$M : \mathcal{A}^\circ \rightarrow \text{Dif } R.$$

More concretely, given objects  $A$  and  $B$  of  $\mathcal{A}$ ,  $M : \mathcal{A}(A, B) \rightarrow \text{Dif } R(MB, MA)$  is a morphism of cochain complexes (it is  $R$ -lineal, homogeneous of degree 0 and commutes with the differentials).

A morphism of  $\mathcal{CA}$  between  $M$  and  $N$  is a natural transformation  $\tau : M \rightarrow N$  such that for each object  $A$  of  $\mathcal{A}$ ,  $\tau_A : MA \rightarrow NA$  is a morphism of cochain complexes.

The category  $\mathcal{CA}$  is an additive category. This additive structure is induced in a natural way from the additivity of  $\text{Dif } R$  and  $Ch^*R$ . Actually,  $\mathcal{CA}$  is an exact category (cf. [K] 2.2).

Denote by  $(Ch^*R, {}_R\mathcal{S}, {}_R\mu, {}_R\lambda)$  the descent structure on the category of cochain complexes of  $R$ -modules  $Ch^*R$  given in section 6.1.

**(6.3.5)** Let  $M = \{M^n, d^i, s^j\}$  be a cosimplicial object of  $\mathcal{CA}$ . Then, for each  $n \geq 0$ , the functor  $M^n : \mathcal{A}^\circ \rightarrow \text{Dif } R$  is a functor of DG-categories. In particular, for a fixed  $A \in \mathcal{CA}$ ,  $M^n A$  is a cochain complex that will be written as  $\{(M^n A)^q, d^{N^n A}\}_{q \in \mathbb{Z}}$ .

On the other hand, the face and degeneracy maps of  $M$  are natural transformations  $d^i : M^n \rightarrow M^{n+1}$ ,  $s^j : M^n \rightarrow M^{n-1}$  satisfying the simplicial identities,

and such that their value at each object  $A$  of  $\mathcal{A}$  is a morphism of cochain complexes

$$d_Z^i : M^n A \rightarrow M^{n+1} A, \quad s_A^j : M^n A \rightarrow M^{n-1} A.$$

Therefore, fixed  $A \in \mathcal{A}$ , it follows that  $MA = \{M^n A, d_A^i, s_A^j\}$  is a cosimplicial cochain complex of  $R$ -modules.

**DEFINITION 6.3.6** (Descent structure on  $\mathcal{CA}$ ).

**Simple functor:** Given  $M \in \Delta\mathcal{CA}$ , the image under  $\mathbf{s}M : \mathcal{A}^\circ \rightarrow \text{Dif } R$  of an object  $A$  of  $\mathcal{A}$  is defined as the cochain complex

$$(\mathbf{s}M)A := {}_R\mathbf{s}(MA) \text{ which is in degree } m((\mathbf{s}M)A)^m = \prod_{p+q=m} (M^p A)^q.$$

If  $f \in \mathcal{A}(A, B)^r$  and  $n \geq 0$  then  $M^n f \in \text{Dif } R(M^n B, M^n A)^r$  is the morphism  $M^n f = \{M^n f^k : (M^n A)^k \rightarrow (M^n A)^{k+r}\}_{k \in \mathbb{Z}}$ , given by

$$(\mathbf{s}M)f = \{((\mathbf{s}M)f)^k\}_{k \in \mathbb{Z}} \text{ where } ((\mathbf{s}M)f)^k = \prod_{p+q=k+r} M^p f^{q-r} : ((\mathbf{s}M)B)^k \rightarrow ((\mathbf{s}M)A)^{k+r}.$$

On the other hand, if  $\tau : M \rightarrow N$  is a morphism of  $\Delta^\circ\mathcal{CA}$  then

$$(\mathbf{s}\tau)_A = {}_R\mathbf{s}(\tau_A) : (\mathbf{s}M)A \rightarrow (\mathbf{s}N)A.$$

**Equivalences:** The class E of equivalences consists of those morphisms  $\rho : M \rightarrow N$  such that  $\rho_A : MA \rightarrow NA$  is a quasi-isomorphism in  $Ch^*R$  for all  $A$  in  $\mathcal{A}$ .

**Transformations  $\lambda$  and  $\mu$ :** Given  $M \in \mathcal{CA}$  and  $Z \in \Delta^\circ\Delta^\circ\mathcal{CA}$ , the transformations  $\lambda(M) : M \rightarrow \mathbf{s}(M \times \Delta)$  and  $\mu(Z) : \mathbf{s}\Delta^\circ\mathbf{s}Z \rightarrow \mathbf{s}DZ$  are define respectively as

$$\lambda(M)_A = {}_R\lambda_{MA} \quad \text{and} \quad \mu(Z)_A = {}_R\mu_{ZA}$$

for each object  $A$  of  $\mathcal{A}$ .

**LEMMA 6.3.7.** *If  $M \in \Delta\mathcal{CA}$ , the mapping  $A \rightarrow (\mathbf{s}M)A = {}_R\mathbf{s}(MA)$  defines a functor*

$$\mathbf{s} : \Delta\mathcal{CA} \rightarrow \mathcal{CA}.$$

*Following the above notations, in addition  $\lambda : Id_{\mathcal{CA}} \rightarrow \mathbf{s}(- \times \Delta)$  and  $\mu : \mathbf{s}\Delta\mathbf{s} \rightarrow \mathbf{s}D$  are in fact natural transformations.*

*Proof.* By (6.3.5)  $MA \in \Delta Ch^*R$ , so  ${}_R\mathbf{s}(MA) \in Ch^*R$  and it is an object of  $\text{Dif } R$ . If  $f : A \rightarrow B$  is a homogeneous morphism of  $\mathcal{A}$  of degree  $r$ , then each



$M^n f : M^n B \rightarrow M^n A$  is an  $R$ -linear morphism, and homogeneous of degree  $r$ . Hence, it is clear that  $(\mathbf{s}M)f : (\mathbf{s}M)B \rightarrow (\mathbf{s}M)A$  is a morphism of  $\text{Dif } R$ . Therefore,  $\mathbf{s}M : \mathcal{A}^\circ \rightarrow \text{Dif } R$  is a functor, and to see that  $\mathbf{s}M$  is an object of  $\mathcal{CA}$  it remains to see that

$$\mathbf{s}M : \mathcal{A}(A, B) \longrightarrow \text{Dif } R((\mathbf{s}M)B, (\mathbf{s}M)A)$$

commutes with the respective boundary maps.

Given  $n \geq 0$ , it holds that  $M^n : \mathcal{A}(A, B) \rightarrow \text{Dif } R(M^n f, M^n A)$  commutes with the boundary maps, that is, if  $f \in \mathcal{A}(A, B)^r$  and  $M^n f = \{M^n f^k : M^n B^k \rightarrow M^n A^{k+r}\}_{k \in \mathbb{Z}}$  then

$$M^n(df) = d(M^n f) = \{d^{M^n A} \circ M^n f^k - (-1)^r M^n f^{k+1} \circ d^{M^n B}\}_{k \in \mathbb{Z}} \in \text{Dif } R(M^n B, M^n A)^{r+1}$$

and  $M^n(df)^k = d^{M^n A} \circ M^n f^k - (-1)^r M^n f^{k+1} \circ d^{M^n B}$ . It follows that

$$((\mathbf{s}M)(df))^k = \prod_{p+q=k+r+1} M^p(df)^{q-r-1} = \prod_{p+q=k+r+1} (d^{M^p A} \circ M^p f^{q-r-1} - (-1)^r M^p f^{q-r} \circ d^{M^p B}) .$$

On the other hand,  $(\mathbf{s}M)f = \{((\mathbf{s}M)f)^k\}_{k \in \mathbb{Z}}$  with  $((\mathbf{s}M)f)^k = \prod_{p+q=k+r} M^p f^{q-r}$ , so

$$d((\mathbf{s}M)f)^k = d^{(\mathbf{s}M)A} \circ \left( \prod_{p+q=k+r} M^p f^{q-r} \right) - (-1)^r \left( \prod_{s+t=k+r+1} M^s f^{t-r-1} \right) \circ d^{(\mathbf{s}M)B} .$$

Set  $\partial^{(M^p B)^q} = \sum_{i=0}^p (-1)^i d_{(M^p B)^q}^i : (M^p B)^q \rightarrow (M^{p+1} B)^q$ , and denote by  $d^{(M^p B)^q} : (M^p B)^q \rightarrow (M^p B)^{q+1}$  the boundary maps of the double complex that is induced by  $MB$ , and similarly for  $MA$ . Note that since  $d^i : M^p \rightarrow M^{p+1}$  is a natural transformation, then  $M^{p+1} f^q \circ d_{(M^p B)^q}^i = d_{(M^p A)^{q+r}}^i \circ M^p f^q$ , and we deduce that  $M^p f^{q-r} \circ \partial^{(M^{p-1} B)^{q-r}} = \partial^{(M^{p-1} A)^q} \circ M^{p-1} f^{q-r}$ .

By definition,  $d^{(\mathbf{s}M)A} : ((\mathbf{s}M)A)^{k+r} \rightarrow ((\mathbf{s}M)A)^{k+r+1}$  and  $d^{(\mathbf{s}M)B} : ((\mathbf{s}M)B)^k \rightarrow ((\mathbf{s}M)B)^{k+1}$  are

$$d^{(\mathbf{s}M)A} = \prod_{p+q=k+r+1} d^{(M^p A)^{q-1}} + (-1)^q \partial^{(M^{p-1} A)^q} ; \quad d^{(\mathbf{s}M)B} = \prod_{s+t=k+1} d^{(M^s B)^{t-1}} + (-1)^t \partial^{(M^{s-1} A)^t}$$

Therefore

$$\begin{aligned} d((\mathbf{s}M)f)^k &= \prod_{p+q=k+r+1} \left( d^{(M^p A)^{q-1}} \circ M^p f^{q-r-1} + (-1)^q \partial^{(M^{p-1} A)^q} \circ M^{p-1} f^{q-r} + \right. \\ &\quad \left. - (-1)^r (M^p f^{q-r} \circ d^{(M^p B)^{q-r-1}} + (-1)^{q-r} M^p f^{q-r} \circ \partial^{(M^{p-1} B)^{q-r}}) \right) = \end{aligned}$$

$$= \prod_{p+q=k+r+1} d^{(M^p A)^{q-1}} \circ M^p f^{q-r-1} - (-1)^r (M^p f^{q-r} \circ d^{(M^p B)^{q-r-1}} = ((sM)(df))^k .$$

Consequently  $sM$  is indeed an object of  $\mathcal{CA}$ . The functoriality of  $s$  with respect to the morphisms of  $\mathcal{CA}$  is clear, so  $s : \Delta\mathcal{CA} \rightarrow \mathcal{CA}$  is a functor as requested.

The naturality of  $\lambda$  and  $\mu$  is a straightforward computation, left to the reader.  $\square$

**PROPOSITION 6.3.8.** *Under the above notations,  $(\mathcal{CA}, s, E, \lambda, \mu)$  is an additive cosimplicial descent category. In addition, the natural transformation  $\mu$  is associative and  $\lambda$  is quasi-invertible.*

*Proof.* We will see that  $(\mathcal{CA}, s, E, \lambda, \mu)$  satisfies the axioms of the notion of cosimplicial descent category.

(CDC 1) is clear. Let us check that the class  $E$  is saturated. Let  $A$  be an object of  $\mathcal{A}$ . By definition of  $\mathcal{CA}$ , the evaluation on  $A$  provides a functor

$$ev_A : \mathcal{CA} \rightarrow Ch^*R \rightarrow HoCh^*R .$$

Moreover, if  $\rho$  is an equivalence in  $\mathcal{CA}$  then  $ev_A(\rho) = \rho_A$  is an isomorphism of  $HoCh^*R$ . Hence, the evaluation functor induces  $ev_A : Ho\mathcal{CA} \rightarrow HoCh^*R$ , which fits into the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{CA} & \xrightarrow{\gamma} & Ho\mathcal{CA} \\ \downarrow ev_A & & \downarrow ev_A \\ Ch^*R & \xrightarrow{\gamma} & HoCh^*R . \end{array}$$

Therefore, if  $\gamma(\rho)$  is an isomorphism of  $Ho\mathcal{CA}$  then  $ev_A(\gamma(\rho)) = \gamma(\rho_A)$  is an isomorphism of  $HoCh^*R$  for each  $A \in \mathcal{CA}$ . Thus  $\rho_A$  is a quasi-isomorphism for each  $A$ , and this means that  $\rho \in E$ . It is also clear that  $E$  is closed by products, so (CDC 2) holds, as well as (CDC 3).

Given  $M \in \mathcal{A}$  and  $Z \in \Delta^\circ \Delta^\circ \mathcal{A}$ , the transformations  $\lambda(M) : s(M \times \Delta) \rightarrow M$  and  $\mu(Z) : s\Delta^\circ sZ \rightarrow sDZ$  are equivalences since  ${}_R\lambda_{MA}$  and  ${}_R\mu_{ZA}$  are quasi-isomorphisms for each object  $A$  of  $\mathcal{A}$ , because  $Ch^*R$  is a cosimplicial descent category.

Thus (CDC 4), (CDC 5) hold, and we can argue similarly for (CDC 6).

Given a morphism  $\rho : M \rightarrow N$  of  $\Delta\mathcal{CA}$ , to see (CDC 7) it is enough to note that  $[C(\rho)](A) = C(\rho_A)$  in  $\Delta^\circ Ch^*R$ .

Finally, (CDC 8) follows from the equality  $(s\Upsilon\rho)(A) = {}_R s(\Upsilon\rho_A)$ .  $\square$

**REMARK 6.3.9.** Given  $M \in \mathcal{CA}$ , we denote by  $[-1] : \mathcal{CA} \rightarrow \mathcal{CA}$  the usual shift functor  $[-1] : Ch^*R \rightarrow Ch^*R$ .

On the other hand, we have the shift functor induced by the descent structure on  $\mathcal{CA}$ , that is  $SM = \mathbf{sPath}(0 \rightarrow M \leftarrow 0)$ .

If  $A \in \mathcal{A}$  then  $(SM)A = {}_R\mathbf{sPath}(0 \rightarrow MA \leftarrow 0)$  and the inclusion  $(MA)[-1] = \mathbf{path}_r(0 \rightarrow MA \leftarrow 0) = {}_R\mathbf{sNPath}(0 \rightarrow MA \leftarrow 0)$  into  $(SM)A$  is a natural homotopy equivalence (see 6.1.5 and 6.1.6).

Then the functors  $S, [-1] : Ho\mathcal{CA} \rightarrow Ho\mathcal{CA}$  are isomorphic, so  $S : Ho\mathcal{CA} \rightarrow Ho\mathcal{CA}$  is an isomorphism of categories. Hence, we obtain from the dual of theorem 4.1.17 the (well-known) triangulated structure on  $Ho\mathcal{CA}$ .

## 6.4 Filtered cochain complexes

Given an abelian category  $\mathcal{A}$ , let  $CF^+\mathcal{A}$  be the category of filtered positive cochain complexes, filtered by a biregular filtration. In this section we will endow  $CF^+\mathcal{A}$  with two different descent structures, whose equivalences will be the filtered quasi-isomorphisms on one hand, and  $E_2$ -isomorphism on the other hand. Both structures will be related through the “decalage” functor of a filtered complex [DeII].

The category of positive cochain complexes will be written as  $Ch^+\mathcal{A}$ , whose objects are complexes  $\{X^n, d\}$  such that  $X^n = 0$  if  $n < 0$ .

**DEFINITION 6.4.1.** A (decreasing) *filtration*  $F$  of an object  $K$  of  $\mathcal{A}$  is a family  $\{F^k K\}_{k \in \mathbb{Z}}$  of subobjects of  $K$  such that  $F^k K \subseteq F^l K$  if  $l \leq k$ .

Denote by  $F\mathcal{A}$  the additive category whose objects are pairs  $(K, F)$  consisting of an object  $K$  of  $\mathcal{A}$  together with a filtration  $F$  of  $K$ , and whose morphisms are those morphisms of  $\mathcal{A}$  compatible with the filtrations.

A filtration  $F$  is said to be *finite* if there exists integers  $n, m \in \mathbb{Z}$  such that  $F^n K = K$  and  $F^m K = 0$ .

The full subcategory of  $F\mathcal{A}$  whose objects are the complexes filtered by a finite filtration will be denoted by  $F_f\mathcal{A}$ . Of course, this is an additive category as well.

**REMARK 6.4.2.** We can consider similarly increasing filtrations instead of decreasing ones. If  $F$  is a decreasing filtration, the convention  $F_k K = F^{-k} K$  allows us to reduce our study to the case of decreasing filtrations.

**DEFINITION 6.4.3.** Let  $\text{CF}^+\mathcal{A}$  be the additive category of pairs  $(A, F)$ , where  $A$  is a filtered positive cochain complex and  $F$  is a biregular decreasing filtration of  $A$ . In other words,  $F = \{F^k A\}_{k \in \mathbb{Z}}$  is such that

1.  $F^k A$  is a subcomplex of  $A \forall k$  and  $A = \cup_k F^k A$ .
2.  $F^{k+1} A \subseteq F^k A \forall k$ .
3. Given  $q \geq 0$ , the filtration  $\{(F^k A)^q\}_k$  of  $A^q$  is finite. Then, there exists integers  $a$  and  $b$  such that  $(F^a A)^q = A^q$  and  $(F^b A)^q = 0$ .

a morphism  $f : (A, F) \rightarrow (B, G)$  of  $\text{CF}^+\mathcal{A}$  is a morphism  $f : A \rightarrow B$  of cochain complexes such that  $f(F^k A) \subseteq G^k B$ , for all  $k$ .

**REMARK 6.4.4.** Equivalently,  $\text{CF}^+\mathcal{A}$  is the category of positive cochain complexes of the additive category  $F_f \mathcal{A}$ .

#### 6.4.1 Filtered quasi-isomorphisms

**DEFINITION 6.4.5.** For each  $k \in \mathbb{Z}$ , the *graded* functor  $\mathbf{Gr}_k : \text{CF}^+\mathcal{A} \rightarrow Ch^+\mathcal{A}$  is defined as

$${}_F \mathbf{Gr}_k A = F^k A / F^{k+1} A \quad .$$

for a filtered cochain complex  $(A, F)$ . A morphism  $f$  of  $\text{CF}^+\mathcal{A}$  is a filtered quasi-isomorphism if  $\mathbf{Gr}_k(f)$  is a quasi-isomorphism for all  $k \in \mathbb{Z}$ .

Let  $(Ch^+\mathcal{A})^{\mathbb{Z}}$  be the category of graded cochain complexes, whose objects are families indexed over  $\mathbb{Z}$  of positive cochain complexes.

The *graded* functor  $\mathbf{Gr} : \text{CF}^+\mathcal{A} \rightarrow (Ch^+\mathcal{A})^{\mathbb{Z}}$  applied to  $(A, F)$  is in degree  $k$  the complex  ${}_F \mathbf{Gr}_k A$ .

**DEFINITION 6.4.6** (Descent structure on  $\text{CF}^+\mathcal{A}$ ).

- Let  $\mathbf{s} : \Delta Ch^+\mathcal{A} \rightarrow Ch^+\mathcal{A}$  be the simple introduced in 6.1. Given  $(A, F) \in \Delta \text{CF}^+\mathcal{A}$  denote by  $\mathbf{s}(F)$  the filtration of  $\mathbf{s}(A)$  defined as  $(\mathbf{s}(F))^k(\mathbf{s}A) = \mathbf{s}(F^k A)$ . The simple functor  $(\mathbf{s}, \mathbf{s}) : \Delta \text{CF}^+\mathcal{A} \rightarrow \text{CF}^+\mathcal{A}$  is given by  $(\mathbf{s}, \mathbf{s})(A, F) = (\mathbf{s}(A), \mathbf{s}(F))$ .
- The class  $E$  consists of the filtered quasi-isomorphisms.
- The natural transformations  $\lambda$  and  $\mu$  in  $\text{CF}^+\mathcal{A}$  are the same as in the cochain complex case.

As well as in the cubical case, [GN] 1.7.5, it holds the following proposition.

**PROPOSITION 6.4.7.** *Under the notations introduced in 6.4.6,  $(\mathbf{CF}^+ \mathcal{A}, (\mathbf{s}, \mathbf{s}), \mathbf{E}, \lambda, \mu)$  is an additive cosimplicial descent category. In addition,  $\mu$  is associative, and  $\lambda$  is quasi-invertible.*

*Proof.* Firstly, note that if  $\mathbb{Z}$  is the discrete category whose objects are the integers and whose morphisms are the identities, then  $(Ch^+ \mathcal{A})^{\mathbb{Z}}$  is just the category of functors from  $\mathbb{Z}$  with values in  $Ch^+ \mathcal{A}$ . By 2.1.15,  $(Ch^+ \mathcal{A})^{\mathbb{Z}}$  is a cosimplicial descent category, with the simple functor induced degreewise, and with equivalences those morphisms that are degreewise quasi-isomorphisms. Let us see that proposition 2.5.8<sup>op</sup> holds for  $\mathbf{Gr} : \mathbf{CF}^+ \mathcal{A} \rightarrow (Ch^+ \mathcal{A})^{\mathbb{Z}}$ .

(SDC 1)<sup>op</sup> is clear since  $\mathbf{CF}^+ \mathcal{A}$  is additive. To see (SDC 3)<sup>op</sup>, we will check that for each  $(A, F)$  in  $\Delta \mathbf{CF}^+ \mathcal{A}$ ,  $\mathbf{s}(F)$  is biregular. Denote  $A = \{A^{nm}\}_{n,m}$  where  $n$  is the cosimplicial degree and  $m$  is the degree relative to  $Ch^+ \mathcal{A}$ .

Fixed  $k \in \mathbb{Z}$ , the complex  $(\mathbf{s}(F))^k(\mathbf{s}(A))$  is in degree  $q$   $\mathbf{s}(F^k A)^q = \bigoplus_{i+j=q} F^k A^{i,j}$ . By assumption  $F$  is biregular on each  $A^n$ , so given  $p \geq 0$  there exists  $a = a(n, p)$  and  $b = b(n, p)$  with  $F^a A^{n,p} = A^{n,p}$  and  $F^b A^{n,p} = 0$ . Let  $\alpha = \alpha(q) = \min\{a(i, j) \mid i + j = q; i, j \geq 0\}$  and  $\beta = \beta(q) = \max\{b(i, j) \mid i + j = q; i, j \geq 0\}$ . Then  $\mathbf{s}(F^\alpha A)^q = \mathbf{s}(A)^q$  and  $\mathbf{s}(F^\beta A)^q = 0$ , so  $\mathbf{s}(F)$  is biregular.

To see (SDC 4)<sup>op</sup> and (SDC 5)<sup>op</sup>, let us check that the transformations  $\mu$  and  $\lambda$  of 6.1 are indeed morphism in  $\mathbf{CF}^+ \mathcal{A}$ .

If  $(A, F) \in \mathbf{CF}^+ \mathcal{A}$ , then  $\mathbf{s}(A \times \Delta)^n = A^n \oplus A^{n-1} \oplus \dots \oplus A^0$  and  $\lambda(A) : A \rightarrow \mathbf{s}(A \times \Delta)$  is the inclusion. Therefore

$$(\lambda(A))(F^k A^n) = F^k A^n \subseteq (\mathbf{s}(F))^k(\mathbf{s}(A \times \Delta)^n) = F^k A^n \oplus F^k A^{n-1} \oplus \dots \oplus F^k A^0.$$

On the other hand, if  $(Z, F) \in \Delta \Delta \mathbf{CF}^+ \mathcal{A}$ , the restriction of  $\mu(Z) : \bigoplus_{i+j+q=n} Z^{i,j,q} \rightarrow \bigoplus_{p+q=n} Z^{p,p,q}$  to  $Z^{i,j,q}$  is  $Z(d^0 \dots d^j, d^p d^{p-1} \dots d^{j+1})$ , where  $p = i + j$ .

Moreover  $(\mathbf{s} \Delta \mathbf{s} F)^k(\mathbf{s} \Delta \mathbf{s} Z)^n = \bigoplus_{i+j+q=n} F^k Z^{i,j,q}$  and  $(\mathbf{s} \Delta \mathbf{s} F)^k(\mathbf{s} \Delta \mathbf{s} Z)^n = \bigoplus_{p+q=n} F^k Z^{p,p,q}$ .

Thus,  $\mu(Z)$  is morphism in  $\mathbf{CF}^+ \mathcal{A}$  since  $Z(d^0 \dots d^j, d^p d^{p-1} \dots d^{j+1})$  preserves the filtration  $F$  for each  $i, j, q$ .

Secondly, (FD 1)<sup>op</sup> holds because  $\mathbf{Gr}$  is additive. It remains to see (FD 2)<sup>op</sup>. We have that the diagram

$$\begin{array}{ccc} \Delta \mathbf{CF}^+ \mathcal{A} & \xrightarrow{\Delta \mathbf{Gr}} & \Delta (Ch^+ \mathcal{A})^{\mathbb{Z}} \\ (\mathbf{s}, \mathbf{s}) \downarrow & & \downarrow \mathbf{s} \\ \mathbf{CF}^+ \mathcal{A} & \xrightarrow{\mathbf{Gr}} & (Ch^+ \mathcal{A})^{\mathbb{Z}} \end{array}$$

commutes up to canonical isomorphism. Indeed, given  $(A, F) \in \mathbf{CF}^+ \mathcal{A}$ ,  $k \in \mathbb{Z}$

and  $n \geq 0$  it holds that

$$\mathbf{s}(\mathbf{F}) \mathbf{Gr}_k(\mathbf{s}(A)^n) = \frac{\mathbf{s}(\mathbf{F})^k(\mathbf{s}(A)^n)}{\mathbf{s}(\mathbf{F})^{k+1}(\mathbf{s}(A)^n)} = \frac{\bigoplus_{i+j=n} \mathbf{F}^k A^{i,j}}{\bigoplus_{i+j=n} \mathbf{F}^{k+1} A^{i,j}} \simeq \bigoplus_{i+j=n} \frac{\mathbf{F}^k A^{i,j}}{\mathbf{F}^{k+1} A^{i,j}} = \mathbf{s}(\mathbf{F} \mathbf{Gr}_k(A))^n$$

and the boundary maps coincide.

If  $(A, \mathbf{F}) \in \mathbf{CF}^+ \mathcal{A}$ , we have that  $\mathbf{Gr}(\lambda(A))$  corresponds to the inclusion  $\mathbf{F} \mathbf{Gr}_k(A) \rightarrow \mathbf{s}((\mathbf{F} \mathbf{Gr}_k(A) \times \Delta) \simeq \mathbf{s}(\mathbf{F}) \mathbf{Gr}_k(\mathbf{s}(A \times \Delta)))$ .

Finally, if  $(Z, \mathbf{F}) \in \Delta \Delta \mathbf{CF}^+ \mathcal{A}$ , it is clear that  $\mathbf{Gr}_k(\mu(Z))$  corresponds to  $\mu(\mathbf{F} \mathbf{Gr}_k(Z))$  through the isomorphisms

$$\mathbf{s} \Delta \mathbf{s}(\mathbf{F}) \mathbf{Gr}_k(\mathbf{s} \Delta \mathbf{s}(Z)) \simeq \mathbf{s} \Delta \mathbf{s}(\mathbf{F} \mathbf{Gr}_k(Z)) \text{ and } \mathbf{s} \mathbf{D}(\mathbf{F}) \mathbf{Gr}_k(\mathbf{s} \mathbf{D}(Z)) \simeq \mathbf{s} \mathbf{D}(\mathbf{F} \mathbf{Gr}_k(Z)) .$$

□

**REMARK 6.4.8.** By simplicity, we have considered the category  $\mathbf{CF}^+ \mathcal{A}$  of uniformly bounded-below cochain complexes, with uniform bound equal to 0, but the arguments remain valid for any fixed value of the bound. So, if  $k$  is a fixed integer and  $\mathbf{CF}^k \mathcal{A}$  is the category of filtered (by a biregular filtration) cochain complexes  $(A, \mathbf{F})$  such that  $A^n = 0$  when  $n < k$ , then

$$(\mathbf{CF}^k \mathcal{A}, (\mathbf{s}, \mathbf{s}), \mathbf{E}, \lambda, \mu)$$

is an additive cosimplicial descent category, where  $\lambda$  is quasi-invertible and  $\mu$  associative.

**DEFINITION 6.4.9** (Filtered homotopies).

Since  $\mathbf{CF}^+ \mathcal{A} = Ch^+(\mathbf{F}_f \mathcal{A})$ , the homotopy theory of  $\mathbf{CF}^+ \mathcal{A}$  is just the one coming from  $Ch^+ \mathbf{F}_f \mathcal{A}$ .

Then, a *filtered homotopy* between the morphisms  $f, g : (A, \mathbf{F}) \rightarrow (B, \mathbf{G})$  in  $\mathbf{CF}^+ \mathcal{A}$  is a homotopy  $h : A^{i+1} \rightarrow B^i$  that preserve the filtrations (that is,  $h(\mathbf{F}^k(A^{i+1})) \subseteq \mathbf{G}^k(B^i)$ ) and such that it is a usual homotopy between  $f$  and  $g$  (that is,  $d^B \circ h + h \circ d^A = f - g$ ). In this case we will say that  $f$  is homotopic to  $g$  in  $\mathbf{CF}^+ \mathcal{A}$ .

**COROLLARY 6.4.10.** *Given morphisms  $A \xrightarrow{f} B \xleftarrow{g} C$  of filtered cochain complexes, the path object associated with  $f$  and  $g$  is a cochain complex  $path(f, g)$ , functorial in  $(f, g)$ , which satisfies the following properties*

1) *there exists functorial maps in  $\mathbf{CF}^+ \mathcal{A}$*

$$j_A : path(f, g) \rightarrow A \quad j_B : path(f, g) \rightarrow B$$

such that  $j_A$  (resp.  $j_C$ ) is a filtered quasi-isomorphism if and only if  $g$  (resp.  $f$ ) is so.

2) If  $f = g = Id_A$ , then there exists a filtered quasi-isomorphism  $P : A \rightarrow \text{path}(A)$  of  $\text{CF}^+ \mathcal{A}$  such that the composition of  $P$  with the projections  $j_A, j'_A : \text{path}(A) \rightarrow A$  given in 1) is equal to the identity on  $A$ .

3) The following square commutes up to filtered homotopy equivalence

$$\begin{array}{ccc} B & \xleftarrow{f} & A \\ \uparrow g & & \uparrow j_A \\ C & \xleftarrow{j_C} & \text{path}(f, g) \end{array} .$$

**REMARK 6.4.11.** If  $C = 0$ , then  $\text{Path}(f, 0)$  is (up to natural filtered homotopy equivalence) the cochain complex  $c(f)[-1]$ , where  $c(f)$  denotes the classical cone, filtered by the induced filtration by those of  $A$  and  $B$ .

Then, following the classical homotopy theory of  $\text{CF}^+ \mathcal{A}$ ,  $f$  gives rise to the distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{i} c(f) \xrightarrow{p} A[1] .$$

In our setting the morphism  $j_A : \text{Path}(f, 0) \rightarrow A$  corresponds to  $c(f)[-1] \xrightarrow{p[-1]} A$ .

On the other hand, the object  $\text{path}(f, g)$  is homotopic to the complex given in 6.1.5, filtered by the filtration which is induced in a natural way by those of  $A$ ,  $B$  and  $C$ .

The category  $\text{HoCF}^+ \mathcal{A}$  is a subcategory of the usual filtered derived category associated with  $\mathcal{A}$ ,  $\text{DF} \mathcal{A} = \text{CF} \mathcal{A}[\mathbb{E}^{-1}]$  (where the cochain complexes does not need to be bounded).

It is known that the class of equivalences  $\mathbb{E}$  has calculus of fractions in  $KF \mathcal{A}$ , and the description of the filtered derived category deduced of this fact is similar to the one given in the following corollary, obtained using our descent techniques.

**COROLLARY 6.4.12.** *The category  $\text{HoCF}^+ \mathcal{A}$  is additive. A morphism  $F : X \rightarrow Y$  of  $\text{HoCF}^+ \mathcal{A}$  can be represented by a zig-zag in the form*

$$X \xleftarrow{w} T \xrightarrow{f} Y , \text{ } w \text{ is a filtered quasi-isomorphism} .$$

Another zig-zag  $X \xleftarrow{u} S \xrightarrow{g} Y$  represents  $F$  if and only if there exists a

hammock (commutative in  $\text{CF}^+\mathcal{A}$ ), relating both zig-zags, in the form

$$\begin{array}{ccccccc}
 & & X & \xleftarrow{w} & T & \xrightarrow{f} & Y \\
 & \nearrow Id & \uparrow & & \uparrow & & \nwarrow Id \\
 X & \longrightarrow & \tilde{X} & \xleftarrow{\quad} & U & \xrightarrow{h} & \tilde{Y} \longleftarrow Y \\
 & \searrow Id & \downarrow & & \downarrow & & \nearrow Id \\
 & & X & \xleftarrow{u} & S & \xrightarrow{g} & Y
 \end{array}$$

where all maps except  $f$ ,  $g$  and  $h$  are filtered quasi-isomorphisms.

One can proceed similarly as in proposition 5.2.7 to deduce the following corollary.

**COROLLARY 6.4.13.** *Let  $\text{CF}^b\mathcal{A}$  be the category of (non-uniformly) bounded-bellow cochain complexes that are filtered by a biregular filtration. Then the localized category  $\text{DF}^b\mathcal{A}$  of  $\text{CF}^b\mathcal{A}$  with respect to the filtered quasi-isomorphisms is a triangulated category.*

The well-known triangulated structure on  $\text{DF}^b\mathcal{A}$  is usually obtained in the literature as a consequence of the exact structure on  $\text{CFA}$ . However, this triangulated structure can be obtained directly (see [III] p. 271), and this is the approach recover here.

**REMARK 6.4.14.** In the case of unbounded cochain complexes, the simple functor of a biregular filtration is not in general a biregular filtration. This is why we have reduced ourselves to the uniformly bounded-bellow case.

However, we can also apply these techniques in the case of not necessarily regular filtrations, dropping the boundness condition. The same happens with biregular filtrations that are zero outside uniform upper and lower bounds. In other words, let  $\text{CF}_f\mathcal{A}$  be the category whose objects are cochain complexes together with a filtration  $F$  such that

$$0 = F^M A \subseteq F^1 A \subseteq \cdots \subseteq F^0 A = A$$

where  $M$  is a fixed integer. In this case, it can be proved similarly that  $(\text{CF}_f\mathcal{A}, (\mathbf{s}, \mathbf{s}), \mu, \lambda)$  is an additive cosimplicial descent category.

### 6.4.2 $E_2$ -isomorphisms

We recall the definition of the spectral sequence associated with a filtered cochain complex.



**DEFINITION 6.4.15.** Let  $(A, F) \in \text{CF}^+ \mathcal{A}$ , and  $r \geq 0$ ,  $p, q \in \mathbb{Z}$ . Define  $Z_r^{p,q}$ ,  $B_r^{p,q}$  and  $E_r^{p,q}$ , where  $E_r^{p,q}$  is called the spectral sequence associated with the filtration  $F$ , as follows

$$\begin{aligned} Z_r^{p,q} &= \ker \left\{ d : F^p A^{p+q} \rightarrow \frac{A^{p+q}}{F^{p+r} A^{p+q}} \right\} \\ \frac{A^{p+q}}{B_r^{p,q}} &= \text{coker} \left\{ d : F^{p-r+1} A^{p+q-1} \rightarrow \frac{A^{p+q}}{F^{p+1} A^{p+q}} \right\} \\ E_r^{p,q} &= \text{Im} \left\{ Z_r^{p,q} \rightarrow \frac{A^{p+q}}{B_r^{p,q}} \right\} = \frac{Z_r^{p,q}}{B_r^{p,q} \cap Z_r^{p,q}} \end{aligned}$$

The boundary map  $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$  is induced by the one of  $A$ . The equality  $d_r d_r = 0$  holds and

$$E_{r+1}^{p,q} = H(E_r^{p-r, q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r, q-r+1}) . \quad (6.5)$$

**REMARK 6.4.16.**

- a) For  $r = 0$ , it holds that  $E_0^{p,q} = {}_F \mathbf{Gr}_p(A^{p+q})$  and  $d = d_0 : E_0^{p,q} \rightarrow E_0^{p,q+1}$ . Then  $E_0 : \text{CF}^+ \mathcal{A} \rightarrow (Ch\mathcal{A})^{\mathbb{Z}}$ , so in  $E_0^{p,q}$  we have that  $q$  is the degree corresponding to  $Ch\mathcal{A}$  and  $p$  the one corresponding to  $\mathbb{Z}$ .

- b) For  $r = 1$ ,  $E_1^{p,q} \stackrel{*}{=} \frac{d^{-1}(F^{p+1} A^{p+q+1}) \cap F^p A^{p+q}}{d(F^p A^{p+q-1}) + F^{p+1} A^{p+q}}$ , and  $d = d_1 : E_1^{p,q} \rightarrow E_1^{p+1, q}$ .

Therefore  $E_1 : \text{CF}^+ \mathcal{A} \rightarrow (Ch\mathcal{A})^{\mathbb{Z}}$ , in such a way that in  $E_1^{p,q}$ ,  $p$  is the degree corresponding to  $Ch\mathcal{A}$  whereas  $q$  corresponds to  $\mathbb{Z}$ .

- c) By (6.5), a morphism  $f$  of  $\text{CF}^+ \mathcal{A}$  is a filtered quasi-isomorphism if and only if  $E_1(f)$  is an isomorphism.
- d) Similarly,  $E_2^{p,q}(f)$  is an isomorphism for all  $p, q$  if and only if  $E_1(f)$  is a quasi-isomorphism in  $(Ch\mathcal{A})^{\mathbb{Z}}$  (that is, it is a quasi-isomorphism in  $Ch\mathcal{A}$  degreewise). We will refer to such morphisms as  $E_2$ -isomorphisms.

**DEFINITION 6.4.17** (Second descent structure on  $\text{CF}^+ \mathcal{A}$ ).

- The simple functor  $(s, \delta) : \Delta \text{CF}^+ \mathcal{A} \rightarrow \text{CF}^+ \mathcal{A}$  is defined as follows. If  $(A, F) \in \text{CF}^+ \mathcal{A}$ , then  $(s, \delta)(A, F) = (s(A), \delta F)$ , where  $s(A)$  is the usual simple of cochain complexes. On the other hand,  $\delta F$  is the *diagonal filtration*<sup>1</sup> over  $s(A)$ , given by

$$(\delta F)^k(s(A)^n) = \bigoplus_{i+j=n} F^{k-i} A^{i,j} .$$

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<sup>1</sup>The filtration  $\delta F$  is the diagonal filtration of  $sF$  and of the natural filtration  $G$  of  $sA$  given by  $G^q = \bigoplus_{p \leq q} A^{p, \cdot}$ .

- The class of equivalences  $E_2$  consists of the  $E_2$ -isomorphisms.
- The transformations  $\lambda$  and  $\mu$  are the same as in 6.4.6, that is, the same as in the cochain complexes case.

**DEFINITION 6.4.18** (“decalage” functor).

Let  $Dec : CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$  be the functor that maps the filtered complex  $(A, F)$  into the filtered complex  $(A, Dec(F))$ , where  $Dec(F)$  is the “decalage” filtration of  $F$  ([DeII] I.3.3), that is also biregular. This filtration is defined as

$$Dec(F)^p A^n = Z_1^{p+n, -p} = \ker \left\{ d : F^{p+n} A^n \rightarrow \frac{A^{n+1}}{F^{p+n+1} A^{n+1}} \right\} .$$

We recall the following result ([DeII] I.3.4).

**LEMMA 6.4.19.**

Consider  $(A, F) \in CF^+ \mathcal{A}$ .

- i) The sequence of inclusions  $Z_1^{p+n+1, -p-1} \subseteq F^{p+n+1} A^n \subseteq B_1^{p+n, -p} \subseteq Z_1^{p+n, -p}$  induces a natural morphism

$$u^{n,p} : E_0^{p, n-p}(Dec(F)) = \frac{Z_1^{p+n, -p}}{Z_1^{p+n+1, -p-1}} \longrightarrow E_1^{p+n, -p}(F) = \frac{Z_1^{p+n, -p}}{B_1^{p+n, -p}} .$$

- ii) Given  $p$ , the morphisms  $u^{*,p}$  gives rise to a morphism of  $Ch\mathcal{A}$ , natural in  $(A, F)$

$$u(A, F) : E_0^{p, *-p}(Dec(F)) \rightarrow E_1^{p+*, -p} .$$

- iii) The morphism  $u(A, F)$  is a quasi-isomorphism, that is, it induces isomorphism in cohomology.

- iv) Using the equation (6.5) in definition 6.4.15, we have for all  $r \geq 1$  that  $u$  induces an isomorphism of graded complexes

$$E_r(Dec(F)) \xrightarrow{\sim} E_{r+1}(F) .$$

By 6.4.19 iv) for  $r = 1$  and 6.4.16 c) we deduce the following corollary.

**COROLLARY 6.4.20.** A morphism  $f$  of  $CF^+ \mathcal{A}$  is an  $E_2$ -isomorphism if and only if  $Dec(f)$  is a filtered quasi-isomorphism. In other words

$$E_2 = \{f \in CF^+ \mathcal{A} \mid Dec(f) \in E\} .$$

**PROPOSITION 6.4.21.** *Under the notations given in 6.4.17,  $(\text{CF}^+ \mathcal{A}, (\mathbf{s}, \delta), \text{E}_2, \lambda, \mu)$  is an additive cosimplicial descent category. In addition,  $\mu$  is associative and  $\lambda$  is quasi-invertible.*

*Proof.* Having into account 6.4.20, it suffices to prove the transfer lemma 2.5.8<sup>op</sup> for the functor  $\text{Dec} : \text{CF}^+ \mathcal{A} \rightarrow \text{CF}^+ \mathcal{A}$ .

Again  $\text{CF}^+ \mathcal{A}$  is additive, so  $(\text{SDC } 1)^{\text{op}}$  holds. Let us see  $(\text{SDC } 3)^{\text{op}}$ , or equivalently, that given  $(A, F) \in \text{CF}^+ \mathcal{A}$ , the filtration  $\delta F$  of  $\mathbf{s}(A)$  is biregular.

By assumption  $F$  is a biregular of  $A^{i,*}$  for all  $i \geq 0$ . So fixed  $j \geq 0$ , there exists  $a(i, j), b(i, j) \in \mathbb{Z}$  such that  $F^k A^{i,j} = A^{i,j} \ \forall k \leq a(i, j)$  and  $F^k A^{i,j} = 0 \ \forall k \geq b(i, j)$ . Then setting  $\alpha = \min\{a(i, j) + i \mid i + j = q ; i, j \geq 0\}$  and  $\beta = \max\{b(i, j) + i \mid i + j = q ; i, j \geq 0\}$  we have that

$$(\delta F)^\alpha(\mathbf{s}(A)^n) = \bigoplus_{i+j=n} F^{\alpha-i} A^{i,j} = \mathbf{s}(A)^n \quad \text{and} \quad (\delta F)^\beta(\mathbf{s}(A)^n) = \bigoplus_{i+j=n} F^{\beta-i} A^{i,j} = 0.$$

Let us prove  $(\text{SDC } 4)^{\text{op}}$ . If  $(Z, F) \in \Delta \Delta \text{CF}^+ \mathcal{A}$ , in degree  $n$   $\mu(Z) : \mathbf{s} \Delta \mathbf{s}(Z) \rightarrow \mathbf{s} \Delta Z$  is the sum of the morphisms  $\mu(Z)_{i,j,q} = Z(d^0 \dots d^j) : Z^{i,j,q} \rightarrow Z^{p,p,q}$ , where  $p = i + j$  and  $p + q = n$ .

The filtration  $\delta \Delta \delta F$  of  $\mathbf{s} \Delta \mathbf{s}(Z)$  is  $(\delta \Delta \delta F)^k(\mathbf{s} \Delta \mathbf{s}(Z))^n = \bigoplus_{i+j+q=n} F^{k-i,k-j} Z^{i,j,q}$ , whereas  $(\delta \Delta F)^k(\mathbf{s} \Delta(Z))^n = \bigoplus_{p+q=n} F^{k-p,k-p} Z^{p,p,q}$ .

Since  $F$  is a decreasing filtration,  $\mu(Z)_{i,j,q}(F^{k-i,k-j} Z^{i,j,q}) \subseteq F^{k-i,k-j} Z^{p,p,q} \subseteq F^{k-p,k-p} Z^{p,p,q}$ , so  $\mu(Z)$  preserves the filtrations.

Let us see now  $(\text{SDC } 5)^{\text{op}}$ . If  $(A, F) \in \text{CF}^+ \mathcal{A}$ , then  $\lambda(A)^n : A^n \rightarrow \mathbf{s}(A \times \Delta)^n = A^n \oplus A^{n-1} \oplus \dots \oplus A^0$  is the inclusion, and  $(\delta(F \times \Delta))^k(\mathbf{s}(A \times \Delta))^n = F^k A^n \oplus F^{k-1} A^{n-1} \oplus \dots \oplus F^0 A^0$ , then  $\lambda(A)$  preserves the filtrations as well.

It is clear that  $\text{Dec}$  is an additive functor, so  $(\text{FD } 1)^{\text{op}}$  holds. To finish the proof it remains to see  $(\text{FD } 2)^{\text{op}}$ . Let us check the commutativity of the following diagram (see [DeIII] 8.I.16)

$$\begin{array}{ccc} \Delta \text{CF}^+ \mathcal{A} & \xrightarrow{\Delta \text{Dec}} & \Delta \text{CF}^+ \mathcal{A} \\ (\mathbf{s}, \delta) \downarrow & & \downarrow (\mathbf{s}, \mathbf{s}) \\ \text{CF}^+ \mathcal{A} & \xrightarrow{\text{Dec}} & \text{CF}^+ \mathcal{A} \end{array}$$

Let  $(A, F) \in \text{CF}^+ \mathcal{A}$ . By definition

$$\begin{aligned} \text{Dec}(\delta F)^p(\mathbf{s}(A))^n &= {}_{\delta F} Z_1^{p+n,-p} = \ker \left\{ d : (\delta F)^{p+n} \mathbf{s}(A)^n \rightarrow \frac{\mathbf{s}(A)^{n+1}}{(\delta F)^{p+n+1} \mathbf{s}(A)^{n+1}} \right\} = \\ &= \ker \left\{ d : \bigoplus_{i+j=n} F^{p+n-i} A^{i,j} \rightarrow \frac{\bigoplus_{l+s=n+1} A^{l,s}}{\bigoplus_{l+s=n+1} F^{p+n+1-i} A^{l,s}} \right\} \end{aligned}$$

The restriction of the boundary map  $d$  to  $A^{i,j}$  is  $d_{A^i} + \sum_k (-1)^{k+j} \partial^k$ , where  $d_{A^i} : A^{i,j} \rightarrow A^{i,j+1}$  is the boundary map of the complex  $A^i$ , and  $\partial^k : A^{i,j} \rightarrow A^{i+1,j}$  is the  $k$ -th face map of  $A$ .

Since  $\partial^k(\mathbb{F}^{p+n-i} A^{i,j}) \subseteq \mathbb{F}^{p+n-i} A^{i+1,j} = \mathbb{F}^{p+n-(i+1)+1} A^{i+1,j} \subseteq (\delta F)^{p+n+1} \mathbf{s}(A)^{n+1}$ , then the restriction of  $d$  to  $\mathbb{F}^{p+n-i} A^{i,j}$  and modulo  $(\delta F)^{p+n+1} \mathbf{s}(A)^{n+1}$  coincides with  $\bigoplus_i d_{A^i}$ . Thus

$$Dec(\delta F)^p(\mathbf{s}(A))^n = \bigoplus_{i+j=n} \ker \left\{ d_{A^i} : \mathbb{F}^{p+j} A^{i,j} \rightarrow \frac{A^{i,j+1}}{\mathbb{F}^{p+j+1} A^{i,j+1}} \right\} = \bigoplus_{i+j=n} Dec(F)^p(A^{i,j}).$$

Therefore  $Dec(\delta F)^p(\mathbf{s}(A))^n = \mathbf{s}(Dec(F))^p \mathbf{s}(A)^n$  and  $(\mathbf{s}, \mathbf{s}) \Delta Dec = Dec(\mathbf{s}, \delta)$ .

Finally, since the image under  $Dec$  of a morphism of  $CF^+ \mathcal{A}$  is the same morphism between the underlying cochain complexes, it is clear that  $Dec(\lambda(A, F)) = \lambda(A, Dec(F))$  if  $(A, F) \in CF^+ \mathcal{A}$ , and  $Dec(\mu(Z, F)) = \mu(Z, Dec(F))$  if  $(Z, F) \in \Delta \Delta CF^+ \mathcal{A}$ .  $\square$

**COROLLARY 6.4.22.** *If we denote by  ${}_1 CF^+ \mathcal{A}$  the category  $CF^+ \mathcal{A}$  with the descent structure given in 6.4.6 and by  ${}_2 CF^+ \mathcal{A}$  the category  $CF^+ \mathcal{A}$  with the one given in 6.4.17, then  $Dec : {}_2 CF^+ \mathcal{A} \rightarrow {}_1 CF^+ \mathcal{A}$  is a functor of additive descent categories.*

**COROLLARY 6.4.23.** *Given morphisms  $A \xrightarrow{f} B \xleftarrow{g} C$  of filtered cochain complexes, the path object associated with  $f$  and  $g$  is a filtered cochain complex  $path(f, g)$ , which is functorial in  $(f, g)$  and such that satisfies the following properties*

1) *there exists functorial maps in  $CF^+ \mathcal{A}$*

$$j_A : path(f, g) \rightarrow A \quad j_B : path(f, g) \rightarrow B$$

*such that  $j_A$  (resp.  $j_B$ ) is an  $E_2$ -isomorphism if and only if  $g$  (resp.  $f$ ) is so.*

2) *If  $f = g = Id_A$ , there exists an  $E_2$ -isomorphism  $P : A \rightarrow path(A)$  of  $CF^+ \mathcal{A}$  such that the composition of  $P$  with the projections  $j_A, j'_A : path(A) \rightarrow A$  given in 1) is equal to the identity on  $A$ .*

3) *The following square commutes up to  $E_2$ -isomorphism*

$$\begin{array}{ccc} B & \xleftarrow{f} & A \\ \uparrow g & & \uparrow j_A \\ C & \xleftarrow{j_C} & path(f, g) \end{array} .$$

**REMARK 6.4.24.** The underlying cochain complex of  $path(f, g)$  coincides with one of the path object given in proposition 6.4.23, but they are not the same object of  $CF^+ \mathcal{A}$ , since now the filtration of  $s(Path(f \times \Delta, g \times \Delta))$  is the diagonal filtration.

**COROLLARY 6.4.25.** *The category  $Ho_2 CF^+ \mathcal{A} = CF^+ \mathcal{A}[E_2^{-1}]$  is additive. A morphism  $F : X \rightarrow Y$  of  $HoCF^+ \mathcal{A}$  is represented by a zig-zag in the form*

$$X \xleftarrow{w} T \xrightarrow{f} Y, \quad w \text{ is an } E_2\text{-isomorphism}.$$

Another zig-zag  $X \xleftarrow{u} S \xrightarrow{g} Y$  represents  $F$  if and only if there exists a hammock (commuting in  $CF^+ \mathcal{A}$ ) relating both zig-zags, in the form

$$\begin{array}{ccccccc} & & X & \xleftarrow{w} & T & \xrightarrow{f} & Y \\ & \nearrow Id & \uparrow & & \uparrow & & \uparrow Id \\ X & \longrightarrow & \tilde{X} & \longleftarrow & U & \xrightarrow{h} & \tilde{Y} \longleftarrow Y \\ & \searrow Id & \downarrow & & \downarrow & & \downarrow Id \\ & & X & \xleftarrow{u} & S & \xrightarrow{g} & Y \end{array},$$

where all maps except  $f, g$  and  $h$  are  $E_2$ -isomorphisms.

Again, one can proceed as in proposition 5.2.7 to deduce the following

**COROLLARY 6.4.26.** *Let  $CF^b \mathcal{A}$  be the category of bounded-below cochain complexes, filtered by a biregular filtration. Then the category localized category  $CF^b \mathcal{A}[E_2^{-1}]$  of  $CF^b \mathcal{A}$  with respect to the  $E_2$ -isomorphisms is a triangulated category.*

*In addition, the “decalage” functor induces a functor of triangulated categories*

$$Dec : D^b F \mathcal{A} \rightarrow CF^b \mathcal{A}[E_2^{-1}].$$

All the results given in this section are satisfied in the case of decreasing filtrations instead of decreasing ones. In particular the following proposition holds.

**PROPOSITION 6.4.27.** *If  $CF^+ \mathcal{A}$  denotes the category of cochain complexes filtered by a biregular increasing filtration, then  ${}_2 CF^+ \mathcal{A} = (CF^+ \mathcal{A}, (s, \delta), E_2, \mu, \lambda)$  is an additive cosimplicial descent category. The diagonal filtration  $\delta$  of the simple of a cosimplicial filtered cochain complex  $(A, W)$  is defined this time as*

$$(\delta W)_k(s(A)^n) = \bigoplus_{i+j=n} W_{k+i} A^{i,j}.$$

In addition, the functor  $Dec : {}_2CF^+\mathcal{A} \rightarrow {}_1CF^+\mathcal{A}$  is a functor of additive cosimplicial descent categories.

## 6.5 Mixed Hodge Complexes

In [DeIII] it is introduced the notion of mixed Hodge complex. The morphisms between such complexes are not given explicitly, but it can be understood that they are those morphisms living in the respective (bi)filtered derived categories that are compatible with the structural morphisms of the mixed Hodge complexes involved.

We will define in this section a category of mixed Hodge complexes, and we will endow it with a structure of cosimplicial descent category using the simple functor developed in [DeIII]. The homotopy category associated with this cosimplicial descent category is mapped into the “category” appearing in loc. cit..

From now on  $\mathcal{A}$  will denote an abelian category. Before giving the notion of mixed Hodge complex, we need to introduce the following preliminaries.

**DEFINITION 6.5.1.** Given a filtered complex  $(A, W)$  of  $CF^+\mathcal{A}$ , the boundary map  $d : A^i \rightarrow A^{i+1}$  is just a morphism of  $F_f\mathcal{A}$  (see 6.4.1). Then,  $d$  is said to be *strictly compatible with the filtration*  $W$  if the morphism

$$A^i / \ker(d) \longrightarrow \text{Im}(d)$$

induced by  $d$  is an isomorphism in  $F_f\mathcal{A}$ , where  $A^i / \ker(d)$  and  $\text{Im}(d)$  are endowed with the filtrations induced by  $W$  (cf. [DeII][I.1]).

**REMARK 6.5.2.** The boundary map of a filtered complex  $(A, W)$  is compatible with the filtration if and only if the spectral sequence associated with  $W$  degenerates at  $E_1$  [DeII][I.3.2].

**DEFINITION 6.5.3** (bifiltered complexes).

Denote by  $CF_2^+\mathcal{A}$  the category whose objects are triples  $(K, W, F)$ , where

- 1.-  $K$  is a positive cochain complex.
- 2.-  $W$  is an increasing biregular filtration of  $K$  (see 6.4.3).
- 3.-  $F$  is a decreasing biregular filtration.

A morphism  $f : (K, W, F) \rightarrow (K', W', F')$  of  $CF_2^+\mathcal{A}$  is a morphism of cochain complexes  $f : K \rightarrow K'$  such that  $f : (K, W) \rightarrow (K', W')$  and  $f : (K, F) \rightarrow (K', F')$  are morphisms of filtered complexes.

(6.5.4) Let  $k$  be a field and  $\mathcal{A}$  be the category of  $k$ -vector spaces.

In order to relax the notations, we will write  $Ch^+k$  instead of  $Ch^+\mathcal{A}$ ,  $CF^+k$  instead of  $CF^+\mathcal{A}$  and  $CF_2^+k$  instead of  $CF_2^+\mathcal{A}$ .

**DEFINITION 6.5.5** (Hodge complex of weight  $n$ ).

A *Hodge complex of weight  $n$*  is the data  $(K_{\mathbb{Q}}, (K_{\mathbb{C}}, F), \alpha)$ , consisting of

- a) A complex  $K_{\mathbb{Q}}$  of  $Ch^+\mathbb{Q}$  such that its cohomology  $H^k K_{\mathbb{Q}}$  has finite dimension over  $\mathbb{Q}$ , for all  $k$ .
- b) A filtered complex  $(K_{\mathbb{C}}, F)$  in  $CF^+\mathbb{C}$ .
- c)  $\alpha$  is  $(\alpha_0, \alpha_1, \tilde{K})$ , where  $\tilde{K}$  is in  $Ch^+\mathbb{C}$  and  $\alpha_i, i = 0, 1$ , are quasi-isomorphisms

$$K_{\mathbb{C}} \xleftarrow{\alpha_0} \tilde{K} \xrightarrow{\alpha_1} K_{\mathbb{Q}} \otimes \mathbb{C} .$$

In addition, the following properties must be satisfied

- (HCI) The boundary map of  $K_{\mathbb{C}}$  is strictly compatible with  $F$ .
- (HCII) For any  $k$ , the filtration over  $H^k(K_{\mathbb{C}}) = H^k(K_{\mathbb{Q}}) \otimes \mathbb{C}$  induced by  $F$  defines a Hodge structure on  $H^k(K_{\mathbb{Q}})$  of weight  $n + k$ .

**REMARK 6.5.6.** (HCII) means that the filtration  $F$  of  $H^k(K_{\mathbb{C}})$  is  $n+k$ -opposite to its conjugate  $\bar{F}$ , that is,  $H^k K_{\mathbb{C}}$  admits the following decomposition into a direct sum

$$H^k K_{\mathbb{C}} = \bigoplus_{p+q=n+k} H^{p,q} \quad \text{where} \quad F^m(H^k K_{\mathbb{C}}) = \bigoplus_{p \geq m} H^{p,q} \quad \text{and} \quad \bar{F}^m(H^k K_{\mathbb{C}}) = \bigoplus_{q \geq m} H^{p,q}$$

or equivalently, [DeII][I.2.5]

$${}_F \mathbf{Gr}_p(\bar{F} \mathbf{Gr}_q(H^k K_{\mathbb{C}})) = 0 \quad \text{if} \quad p + q \neq n + k .$$

In the previous definition the zig-zag  $\alpha$  is in the form  $\cdot \leftarrow \cdot \rightarrow \cdot$ , but we can consider as well any other kind of zig-zag relating  $K_{\mathbb{C}}$  and  $K_{\mathbb{Q}} \otimes \mathbb{C}$ .

**DEFINITION 6.5.7.** Let  $\mathcal{A}$  be a category endowed with a class of morphisms  $\mathcal{W}$ . A  $\mathcal{W}$ -zig-zag of  $\mathcal{A}$  (or just zig-zag, if  $\mathcal{W}$  is understood) is a pair  $(\underline{A}, \underline{w})$  consisting of a family  $\underline{A} = \{A_0, \dots, A_r\}$  of objects of  $\mathcal{A}$  together with morphisms  $\underline{w} = \{w_0, \dots, w_{r-1}\}$  of  $\mathcal{W}$ , such that each  $w_i$  is a morphism between  $A_i$  and  $A_{i+1}$  (that is, either  $w_i : A_i \rightarrow A_{i+1}$  or  $w_i : A_{i+1} \rightarrow A_i$ ).

In addition, we will assume that two consecutive arrows have opposite senses,

that is, either  $A_{i-1} \xrightarrow{w_{i-1}} A_i \xleftarrow{w_i} A_{i+1}$  or  $A_{i-1} \xleftarrow{w_{i-1}} A_i \xrightarrow{w_i} A_{i+1}$ .

Two  $\mathcal{W}$ -zig-zags  $(\underline{A}, \underline{w})$ ,  $(\underline{B}, \underline{v})$  are said to be of the *same kind* if their respective families of objects have the same cardinality  $r$  and if each  $w_i$  has the same sense as  $v_i$ , for  $i = 0, \dots, r-1$ .

A morphism between two  $\mathcal{W}$ -zig-zags  $(\underline{A}, \underline{w})$ ,  $(\underline{B}, \underline{v})$  of the same kind is a family of morphisms  $\underline{f} = \{f_i : A_i \rightarrow B_i\}_i$  such that each diagram involving the maps  $f$ ,  $w$  and  $v$  is commutative.

A  $\mathcal{W}$ -zig-zag between  $A$  and  $B$  is just a  $\mathcal{W}$ -zig-zag  $(\underline{A}, \underline{w})$  such that  $A_0 = A$  and  $A_r = B$ .

Given objects  $A$  and  $B$  of  $\mathcal{A}$ , the  $\mathcal{W}$ -zig-zags between  $A$  and  $B$  are the objects of a category, that will be denoted by  $\mathcal{Rist}^{\mathcal{W}}(A, B)$ . The  $\mathcal{W}$ -zig-zags between  $A$  and  $B$  of the same kind as  $\cdot \leftarrow \cdot \rightarrow \cdot$  gives rise to the full subcategory of  $\mathcal{Rist}^{\mathcal{W}}(A, B)$ , that will be denoted by  $\mathcal{Rist}_{red}^{\mathcal{W}}(A, B)$ .

In addition, if we invert the morphisms of  $\mathcal{W}$  in  $\mathcal{A}$ , then each  $\mathcal{W}$ -zig-zag becomes a morphism, so we have the functors

$$\begin{aligned} \mathcal{Rist}_{red}(A, B) &\xrightarrow{\gamma} \text{Hom}_{\mathcal{A}[\mathcal{W}^{-1}]}(A, B) & \mathcal{Rist}^{\mathcal{W}}(A, B) &\xrightarrow{\gamma} \text{Hom}_{\mathcal{A}[\mathcal{W}^{-1}]}(A, B) \\ \mathcal{Rist}_{red} &\xrightarrow{\gamma} Fl(\mathcal{A}[\mathcal{W}^{-1}]) & \mathcal{Rist}^{\mathcal{W}} &\xrightarrow{\gamma} Fl(\mathcal{A}[\mathcal{W}^{-1}]) . \end{aligned}$$

**(6.5.8)** Let **Quis** be the class of quasi-isomorphisms of  $Ch^+k$ , and **QuisF** be the class of filtered quasi-isomorphisms of  $CF^+k$ , where  $k = \mathbb{Q}, \mathbb{C}$ .

Then, the data  $\alpha$  of a Hodge complex of weight  $n$  is just an object of the category  $\mathcal{Rist}_{red}^{\mathbf{Quis}}(K_{\mathbb{C}}, K_{\mathbb{Q}} \otimes \mathbb{C})$ .

**DEFINITION 6.5.9.** A *generalized Hodge complex of weight  $n$*  consists of  $(K_{\mathbb{Q}}, (K_{\mathbb{C}}, F), \alpha)$ , where  $K_{\mathbb{Q}}$  and  $(K_{\mathbb{C}}, F)$  satisfies conditions a), b) (HCI) and (HCII) in definition 6.5.5, whereas  $\alpha$  is a **Quis**-zig-zag between  $K_{\mathbb{C}}$  and  $K_{\mathbb{Q}} \otimes \mathbb{C}$  of  $Ch^+\mathbb{C}$ .

Due to properties 1) and 3) in proposition 6.1.3, we can associate a Hodge complex of weight  $n$  to a generalized Hodge complex of weight  $n$  in a functorial way.

**LEMMA 6.5.10.** *If  $\mathcal{A}$  is an abelian category, there exists a functor*

$$\underline{red} : \mathcal{Rist}^{\mathbf{Quis}} \longrightarrow \mathcal{Rist}_{red}^{\mathbf{Quis}}$$

*such that the composition  $\mathcal{Rist}^{\mathbf{Quis}} \xrightarrow{\underline{red}} \mathcal{Rist}_{red}^{\mathbf{Quis}} \xrightarrow{\gamma} Fl(HoCh^+\mathcal{A})$  is just  $\gamma : \mathcal{Rist}^{\mathbf{Quis}} \longrightarrow Fl(HoCh^+\mathcal{A})$ .*



In addition, if  $A, B$  are objects of  $Ch^+ \mathcal{A}$ , the functor  $\underline{red}$  restrict to

$$\underline{red} : \mathcal{Rist}^{\mathbf{Quis}}(A, B) \longrightarrow \mathcal{Rist}_{red}^{\mathbf{Quis}}(A, B)$$

We will say that  $\underline{red}(\underline{A}, \underline{w})$  is the reduced zig-zag associated with  $(\underline{A}, \underline{w})$ .

*Proof.* Given a **Quis**-zig-zag  $R = (\underline{A}, \underline{w})$  between  $A$  and  $B$  in the form

$$A = A_0 \xrightarrow{w_0} A_1 \xrightarrow{w_1} \cdots \xrightarrow{w_{r-2}} A_{r-1} \xrightarrow{w_{r-1}} A_r$$

its associated reduced zig-zag is obtained through the following procedure.

We take the first pair of consecutive arrows in  $R$  of the form  $A_{i-1} \xrightarrow{w_{i-1}} A_i \xleftarrow{w_i} A_{i+1}$ . If there is no such pair of consecutive arrows in  $R$ , then this zig-zag is already a zig-zag in  $\mathcal{Rist}_{red}^{\mathbf{Quis}}(A, B)$ , and we define  $\underline{red}(R) = R$ .

If there exists such  $w_i, w_{i-1}$ , properties 1) and 2) of proposition 6.1.3 provide the following square, commutative up to homotopy,

$$\begin{array}{ccc} A_i & \xleftarrow{w_i} & A_{i+1} \\ \uparrow w_{i-1} & & \uparrow J_{A_{i+1}} \\ A_{i-1} & \xleftarrow{J_{A_{i-1}}} path(w_i, w_{i-1}) & \end{array},$$

where  $J_{A_{i+1}}$  and  $J_{A_{i-1}}$  are quasi-isomorphisms, that is, morphisms of **Quis**.

Hence, replacing in  $R$  the maps  $A_{i-1} \xrightarrow{w_{i-1}} A_i \xleftarrow{w_i} A_{i+1}$  with  $A_{i-1} \xleftarrow{J_{A_{i-1}}} path(w_i, w_{i-1}) \xrightarrow{J_{A_{i+1}}} A_{i+1}$  and composing maps we obtain a new **Quis**-zig-zag  $\tilde{R}$  between  $A$  and  $B$  of length strictly smaller than the length of  $R$ .

Since  $w_i \circ J_{A_{i+1}}$  is homotopic to  $w_{i-1} \circ J_{A_{i-1}}$ , then  $\gamma(R)$  and  $\gamma(\tilde{R})$  coincides in  $HoCh^+ \mathcal{A}$ .

Moreover, the mapping  $R \rightarrow \tilde{R}$  defines a functor  $\mathcal{Rist}^{\mathbf{Quis}}(A, B) \rightarrow \mathcal{Rist}^{\mathbf{Quis}}(A, B)$ .

Indeed, if we have a commutative diagram in  $Ch^+ \mathcal{A}$

$$\begin{array}{ccccc} A_{i-1} & \xrightarrow{w_{i-1}} & A_i & \xleftarrow{w_i} & A_{i+1} \\ f_{i-1} \downarrow & & f_i \downarrow & & f_{i+1} \downarrow \\ B_{i-1} & \xrightarrow{v_{i-1}} & B_i & \xleftarrow{v_i} & B_{i+1} \end{array}$$

then from the functoriality of  $path$  it follows the existence of a morphism  $\tilde{f}$  that fits into the commutative diagram

$$\begin{array}{ccccc} A_{i-1} & \xleftarrow{J_{A_{i+1}}} path(w_i, w_{i-1}) & \xrightarrow{J_{A_{i+1}}} & A_{i+1} \\ f_{i-1} \downarrow & & \tilde{f} \downarrow & & f_{i+1} \downarrow \\ B_{i-1} & \xleftarrow{J_{B_{i+1}}} path(v_i, v_{i-1}) & \xrightarrow{J_{B_{i+1}}} & B_{i+1} \end{array}.$$

Therefore, it suffices to iterate this procedure until we get the desired zig-zag  $\underline{red}(A, \underline{w})$ .  $\square$

**REMARK 6.5.11.** Note that the reduced zig-zag associated with  $(\underline{A}, \underline{w})$  not only preserves the morphism in  $HoCh_*(\mathcal{A})$  represented by this zig-zag. In addition, the original and reduced zig-zags are in some sense “homotopic”.

**COROLLARY 6.5.12.** *Each generalized Hodge complex of weight  $n$  gives rise to a Hodge complex of weight  $n$ , just by replacing  $\alpha$  with  $\underline{red}(\alpha)$ .*

Next we recall the notion of mixed Hodge complex, and introduce a category consisting of these complexes.

**(6.5.13)** The tensor product over  $\mathbb{C}$ ,  $- \otimes \mathbb{C} : \mathbb{Q}\text{-vector spaces} \rightarrow \mathbb{C}\text{-vector spaces}$ , is an exact functor. Thus it induces

$$- \otimes \mathbb{C} : CF^+ \mathbb{Q} \rightarrow CF^+ \mathbb{C}$$

in such a way that the functor  $\mathbf{Gr}_n$  commutes with  $- \otimes \mathbb{C}$ .

**DEFINITION 6.5.14 (Mixed Hodge Complex).**

A *mixed Hodge complex* consists of the data  $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$ , where

- a)  $(K_{\mathbb{Q}}, W)$  is a cochain complex of  $\mathbb{Q}$ -vector spaces, filtered by the increasing filtration  $W$ . In other words,  $(K_{\mathbb{Q}}, W)$  is an object of  $CF^+ \mathbb{Q}$ . In addition,  $H^k K_{\mathbb{Q}}$  has finite dimension over  $\mathbb{Q}$  for all  $k$ .
- b)  $(K_{\mathbb{C}}, W, F)$  is an object of  $CF_2^+ \mathbb{C}$ .
- c)  $\alpha$  is the data  $(\alpha_0, \alpha_1, (\tilde{K}, \tilde{W}))$ , where  $(\tilde{K}, \tilde{W})$  is an object of  $CF^+ \mathbb{C}$  and  $\alpha_i, i = 0, 1$ , is a filtered quasi-isomorphism (see 6.4.5)

$$(K_{\mathbb{C}}, W) \xleftarrow{\alpha_0} (\tilde{K}, \tilde{W}) \xrightarrow{\alpha_1} (K_{\mathbb{Q}}, W) \otimes \mathbb{C} .$$

The following axiom must be satisfied

(MHC) For each  $n$ ,  $({}_W \mathbf{Gr}_n K_{\mathbb{Q}}, ({}_W \mathbf{Gr}_n K_{\mathbb{C}}, F), \mathbf{Gr}_n(\alpha))$  is a Hodge complex of weight  $n$ , where  $\mathbf{Gr}_n(\alpha)$  denotes the zig-zag

$${}_W \mathbf{Gr}_n K_{\mathbb{C}} \xleftarrow{\mathbf{Gr}_n \alpha_0} {}_{\tilde{W}} \mathbf{Gr}_n \tilde{K} \xrightarrow{\mathbf{Gr}_n \alpha_1} {}_{W \otimes \mathbb{C}} \mathbf{Gr}_n (K_{\mathbb{Q}} \otimes \mathbb{C}) \stackrel{(6.5.13)}{\cong} ({}_W \mathbf{Gr}_n K_{\mathbb{Q}}) \otimes \mathbb{C} .$$

**REMARK 6.5.15.** Again, the data  $\alpha$  of a mixed Hodge complex is just an object of the category  $\mathcal{Rist}_{red}^{\mathbf{Quis}}((K_{\mathbb{C}}, W), (K_{\mathbb{Q}}, W) \otimes \mathbb{C})$ , where  $\mathbf{QuisF}$  is the class of the filtered quasi-isomorphisms of  $CF^+ \mathbb{C}$ .

Analogously to the case of Hodge complexes of weight  $n$ , we can consider any kind of zig-zag to define the data  $\alpha$  of a mixed Hodge complex.

**DEFINITION 6.5.16.** A *generalized mixed Hodge complex* consists of  $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$ , where  $(K_{\mathbb{Q}}, W)$  and  $(K_{\mathbb{C}}, W, F)$  satisfies conditions a) and b) of mixed Hodge complex.

As before,  $\alpha$  is a **QuisF**-zig-zag between  $(K_{\mathbb{C}}, W)$  and  $(K_{\mathbb{Q}}, W) \otimes \mathbb{C}$  in  $CF^+\mathbb{C}$ . In addition, the following axiom must be satisfied

(MHC) For each  $n$ ,  $(w\mathbf{Gr}_n K_{\mathbb{Q}}, (w\mathbf{Gr}_n K_{\mathbb{C}}, F), \mathbf{Gr}_n(\alpha))$  is a generalized Hodge complex of weight  $n$  (where  $\mathbf{Gr}_n(\alpha)$  is defined analogously).

Similarly to 6.5.10, properties 1) and 3) of the functor *path* given in proposition 6.4.23 allows us to associate a mixed Hodge complex to any generalized mixed Hodge complex in a functorial way.

**LEMMA 6.5.17.** *If  $\mathcal{A}$  is an abelian category, there exists a functor*

$$\underline{red} : \mathcal{Rist}^{\mathbf{QuisF}} \longrightarrow \mathcal{Rist}_{red}^{\mathbf{QuisF}}$$

*such that the composition  $\mathcal{Rist}^{\mathbf{QuisF}} \xrightarrow{\underline{red}} \mathcal{Rist}_{red}^{\mathbf{QuisF}} \xrightarrow{\gamma} Fl(HoCF^+ \mathcal{A})$  is just  $\gamma : \mathcal{Rist}^{\mathbf{QuisF}} \longrightarrow Fl(HoCF^+ \mathcal{A})$ .*

*In addition, if  $(A, F), (B, G)$  are objects of  $CF^+ \mathcal{A}$ , functor  $\underline{red}$  restricts to*

$$\underline{red} : \mathcal{Rist}^{\mathbf{QuisF}}((A, F), (B, G)) \longrightarrow \mathcal{Rist}_{red}^{\mathbf{QuisF}}((A, F), (B, G))$$

*We will refer to  $\underline{red}((\underline{A}, \underline{F}), \underline{w})$  as the reduced zig-zag associated with  $((\underline{A}, \underline{F}), \underline{w})$ .*

**REMARK 6.5.18.** The square appearing in property 3) of proposition 6.4.23 commutes up to filtered homotopy, so the reduced zig-zag associated with  $((\underline{A}, \underline{F}), \underline{w})$  is “homotopy equivalent” (in some sense) to  $((\underline{A}, \underline{F}), \underline{w})$ .

In addition, this is a constructive procedure, consisting just in iterate functor *path* (in an ordered way).

**COROLLARY 6.5.19.** *Each generalized mixed Hodge complex gives rise to a mixed Hodge complex in a functorial way by replacing  $\alpha$  by  $\underline{red}(\alpha)$ .*

**EXAMPLE 6.5.20.** [DeIII], 8.I.8 Let  $j : U \rightarrow X$  be an open immersion of smooth varieties (here variety means a separated, reduced and of finite type  $\mathbb{C}$ -scheme). Assume that  $X$  is proper and  $Y = X \setminus U$  is a normal crossing divisor. Let  $(Rj_* \mathbb{Q}, W)$  be the filtered complex of sheaves of  $\mathbb{Q}$ -vector spaces on  $X$ , where  $W = \tau_{\leq}$  is the “canonical” filtration. That is to say,  $\tau_{\leq p} Rj_* \mathbb{Q}$  is given in

degree  $n$  by  $Rj_*\mathbb{Q}$  if  $n < p$ ,  $\text{Ker}d$  if  $p = n$  and 0 otherwise.

Let  $(\Omega_X\langle Y\rangle, W, F)$  be the logarithmic De Rham complex of  $X$  along  $Y$  [DeII] 3.I. The filtration  $W$  is the so-called “weight filtration”, consisting in filtering by the order of poles in  $\Omega_X\langle Y\rangle$ . The filtration  $F$ , called “Hodge filtration”, is just the filtration “bête” associated with  $\Omega_X\langle Y\rangle$ , that is,  $F^n\Omega_X^p\langle Y\rangle = \Omega_X^p\langle Y\rangle$  if  $p \geq n$  and 0 otherwise.

Then, there exists a zig-zag  $\alpha$  of filtered quasi-isomorphisms such that

$$(R\Gamma(j_*\mathbb{Q}, W), R\Gamma(\Omega_X\langle Y\rangle, W, F), \alpha)$$

is a mixed Hodge complex.

The zig-zag  $\alpha$  involves the result [DeII] 3.I.8 that relates  $\Omega_X\langle Y\rangle$  to  $j_*\Omega_U$ , together with Poincaré lemma (that is,  $\Omega_U$  is a resolution of the constant sheaf  $\mathbb{C}$ ), and together with Godement resolutions.

The zig-zag  $\alpha = \alpha(U, X)$  is natural in  $(U, X)$ . Moreover, since Godement resolutions are functorial, the natural transformations that gives rise to  $\alpha(U, X)$  given in [DeII] has values in the category of filtered complexes instead of in the filtered derived category (cf. [Be], 4 or [H], 8.2).

It should be pointed out that the zig-zag  $\alpha$  is also considered as a zig-zag of length 2 in [H], using a dual procedure to the one given here, that is called “quasi-pushout” in loc. cit..

**DEFINITION 6.5.21 (Category of mixed Hodge complexes).**

Let  $\mathcal{Hdg}$  be the category whose objects are the mixed Hodge complexes, and whose morphisms are defined as follows.

A morphism  $f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \tilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$  consists of

- 1) A morphism  $f_{\mathbb{Q}} : (K_{\mathbb{Q}}, W) \rightarrow (K'_{\mathbb{Q}}, W')$  of  $\text{CF}^+\mathbb{Q}$ .
- 2) A morphism  $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \rightarrow (K'_{\mathbb{C}}, W', F')$  of  $\text{CF}_2^+\mathbb{C}$ .
- 3) If  $\alpha$  and  $\alpha'$  are the respective zig-zags

$$\begin{aligned} (K_{\mathbb{C}}, W) &\xleftarrow{\alpha_0} (\tilde{K}, \tilde{W}) \xrightarrow{\alpha_1} (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\ (K'_{\mathbb{C}}, W') &\xleftarrow{\alpha'_0} (\tilde{K}', \tilde{W}') \xrightarrow{\alpha'_1} (K'_{\mathbb{Q}}, W') \otimes \mathbb{C} \end{aligned}$$

then  $\tilde{f} : (\tilde{K}, \tilde{W}) \rightarrow (\tilde{K}', \tilde{W}')$  is a morphism of  $\text{CF}_2^+\mathbb{C}$  such that the squares

I and II of diagram

$$\begin{array}{ccccc}
(K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\widetilde{K}, \widetilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\
f_{\mathbb{C}} \downarrow & & \downarrow \widetilde{f} & & \downarrow f_{\mathbb{Q}} \otimes \mathbb{C} \\
(K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\widetilde{K}', \widetilde{W}') & \xrightarrow{\alpha'_1} & (K'_{\mathbb{Q}}, W') \otimes \mathbb{C}
\end{array} \quad \begin{array}{c} \text{I} \qquad \qquad \text{II} \end{array} \quad (6.6)$$

**commutes** in  $\text{CF}_2^+\mathbb{C}$ . For a similar definition see [Be], 3.

**DEFINITION 6.5.22.** The category of generalized mixed Hodge complexes,  $\mathcal{Hdg}_G$ , is defined analogously.

A morphism  $f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \widetilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$  between two generalized mixed Hodge complexes such that  $\alpha$  and  $\alpha'$  are of the same kind, consists of

- 1) A morphism  $f_{\mathbb{Q}} : (K_{\mathbb{Q}}, W) \rightarrow (K'_{\mathbb{Q}}, W')$  of  $\text{CF}^+\mathbb{Q}$ .
- 2) A morphism  $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \rightarrow (K'_{\mathbb{C}}, W', F')$  of  $\text{CF}_2^+\mathbb{C}$ .
- 3) A morphism  $\widetilde{f}$  between  $\alpha$  and  $\alpha'$  in the category  $\mathcal{Rist}^{\mathbf{QuisF}}$  of zig-zags of filtered quasi-isomorphisms in  $\text{CF}^+\mathbb{C}$ .

**COROLLARY 6.5.23.** The functor “reduced zig-zag” gives rise to a functor

$$\underline{red} : \mathcal{Hdg}_G \longrightarrow \mathcal{Hdg}$$

that maps the generalized mixed Hodge complex  $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$  into the mixed Hodge complex  $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \underline{red} \alpha)$ .

Moreover, the zig-zags  $\alpha$  and  $\underline{red} \alpha$  define the same morphism of  $\text{HoCF}^+\mathbb{Q}$  and, in addition, they are “homotopy equivalent”.

*Proof.* The functoriality of  $\underline{red} : \mathcal{Hdg}_G \longrightarrow \mathcal{Hdg}$  is clear.

If  $f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \widetilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$ , is a morphism in  $\mathcal{Hdg}_G$  then  $(f_{\mathbb{Q}}, f_{\mathbb{C}}, \underline{red} \widetilde{f})$  is a morphism in  $\mathcal{Hdg}$ , because of the functoriality of  $\underline{red} : \mathcal{Rist}^{\mathbf{QuisF}} \longrightarrow \mathcal{Rist}_{red}^{\mathbf{QuisF}}$ .  $\square$

**REMARK 6.5.24.** Assume given

$$f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \widetilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$$

such that  $f_{\mathbb{Q}} : (K_{\mathbb{Q}}, W) \rightarrow (K'_{\mathbb{Q}}, W')$  and  $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \rightarrow (K'_{\mathbb{C}}, W', F')$ . Assume also that  $\alpha$  and  $\alpha'$  consists of the respective zig-zags

$$\begin{array}{ccccc}
(K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\widetilde{K}, \widetilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\
(K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\widetilde{K}', \widetilde{W}') & \xrightarrow{\alpha'_1} & (K'_{\mathbb{Q}}, W') \otimes \mathbb{C}
\end{array}$$

and that  $\tilde{f} : (\tilde{K}, \tilde{W}) \rightarrow (\tilde{K}', \tilde{W}')$  is a morphism of  $\mathrm{CF}_2^+\mathbb{C}$  such that the squares I and II of diagram

$$\begin{array}{ccccc}
(K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\
f_{\mathbb{C}} \downarrow & & \tilde{f} \downarrow & & \downarrow f_{\mathbb{Q}} \otimes \mathbb{C} \\
(K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\tilde{K}', \tilde{W}') & \xrightarrow{\alpha'_1} & (K'_{\mathbb{Q}}, W') \otimes \mathbb{C}
\end{array}
\quad \begin{array}{c} \text{I} \qquad \qquad \text{II} \end{array}$$

commutes **up to filtered homotopy** in  $\mathrm{CF}_2^+\mathbb{C}$ .

Let  $\mathrm{cyl}(\tilde{K}, \tilde{W})$  be the “classical” cylinder object in the category  $\mathrm{CF}_2^+\mathbb{C} = C(\mathrm{F}_f\mathbb{C})$  (see 6.4.9), and  $i, j : (\tilde{K}, \tilde{W}) \rightarrow \mathrm{cyl}(\tilde{K}, \tilde{W})$  be the canonical inclusions.

Recall that  $f_{\mathbb{C}} \circ \alpha_0$  is homotopic to  $\alpha'_0 \circ \tilde{f}$  in  $\mathrm{CF}_2^+\mathbb{C}$  if and only if there exists a homotopy  $H : \mathrm{cyl}(\tilde{K}, \tilde{W}) \rightarrow (\tilde{K}, \tilde{W})$  giving rise to the following morphism of **QuisF**-zig-zags in  $\mathrm{CF}_2^+\mathbb{C}$

$$\begin{array}{ccccccc}
(K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{i} & \mathrm{cyl}(\tilde{K}, \tilde{W}) & \xleftarrow{j} & (\tilde{K}, \tilde{W}) \\
f_{\mathbb{C}} \downarrow & & f_{\mathbb{C}} \circ \alpha_0 \downarrow & & H \downarrow & & \tilde{f} \downarrow \\
(K'_{\mathbb{C}}, W') & \xleftarrow{Id} & (K'_{\mathbb{C}}, W') & \xrightarrow{Id} & (K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\tilde{K}', \tilde{W}')
\end{array}$$

One can argue in a similar way with square II, obtaining a morphism of  $\mathcal{H}dg_{\mathcal{G}}$ . Therefore, we obtain in this way a morphism in  $\mathcal{H}dg$  between the corresponding mixed Hodge complexes.

This mapping is not functorial at all in the data  $(f_{\mathbb{Q}}, f_{\mathbb{C}}, \tilde{f})$ , since it depends on chosen the homotopy for the squares I and II.

Two different choices of homotopies for I and II provides two morphisms of  $\mathcal{H}dg$ , that are not related in general.

**REMARK 6.5.25.** In [PS] another definition of category of mixed Hodge complexes is considered, in which a morphism is such that the corresponding diagram (6.6) commutes up to homotopy. In this case some pathologies appear, for instance the non-functoriality of the cone associated with a morphism of mixed Hodge complexes (see loc. cit. 3.23).

Next we endow  $\mathcal{H}dg$  with a structure of cosimplicial descent category, in which the simple functor  $\mathbf{s}_{\mathcal{H}dg} = (\mathbf{s}, \delta, \mathbf{s}) : \Delta\mathcal{H}dg \rightarrow \mathcal{H}dg$  is the one given in [DeIII] 8.I.15.

**REMARK 6.5.26.** Note that the simple functor  $(s, \delta) : \Delta CF^+ \mathbb{Q} \rightarrow CF^+ \mathbb{Q}$  (see 6.4.27) commutes with  $- \otimes \mathbb{C}$ , since the tensor product with  $\mathbb{C}$  commutes with finite sums.

**DEFINITION 6.5.27** (Descent structure on  $\mathcal{H}dg$ ).

**Simple functor:** Given a cosimplicial mixed Hodge complex  $K = ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$ , let  $s_{\mathcal{H}dg} K$  be the mixed Hodge complex  $((sK_{\mathbb{Q}}, \delta W), (sK_{\mathbb{C}}, \delta W, sF), s\alpha)$ , where  $s$  denotes the usual simple of cochain complexes and  $\delta W$  is defined as in 6.4.27. More concretely

$$\begin{aligned} s(K_-)^n &= \bigoplus_{p+q=n} K_-^{p,q} ; \quad (\delta W)_k(s(K_-)^n) = \bigoplus_{i+j=n} W_{k+i} K_-^{i,j} , \quad \text{if } - \text{ is } \mathbb{Q} \text{ or } \mathbb{C} \\ (s(F))^k(sK_{\mathbb{C}})^n &= \bigoplus_{p+q=n} F^k K_{\mathbb{C}}^{p,q} . \end{aligned}$$

Finally, if  $\alpha = (\alpha_0, \alpha_1, (\widetilde{K}, \widetilde{W}))$  then  $s\alpha$  denotes the zig-zag

$$(sK_{\mathbb{C}}, \delta W) \xleftarrow{s\alpha_0} (s\widetilde{K}, \delta\widetilde{W}) \xrightarrow{s\alpha_1} (s(K_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W \otimes \mathbb{C})) \stackrel{6.5.26}{\simeq} (sK_{\mathbb{Q}}, \delta W) \otimes \mathbb{C} . \quad (6.7)$$

**Equivalences:** the class of equivalences is defined as

$$E_{\mathcal{H}dg} = \{(f_{\mathbb{Q}}, f_{\mathbb{C}}, \widetilde{f}) \mid f_{\mathbb{Q}} \text{ is a quasi-isomorphism in } Ch^+ \mathbb{Q}\} .$$

**Transformation  $\lambda$ :**  $\lambda^{\mathcal{H}dg} : Id_{\mathcal{H}dg} \rightarrow s_{\mathcal{H}dg}(- \times \Delta)$  is  $\lambda_K^{\mathcal{H}dg} = (\lambda_{K_{\mathbb{Q}}}^{\mathbb{Q}}, \lambda_{K_{\mathbb{C}}}^{\mathbb{C}}, \lambda_{\widetilde{K}}^{\mathbb{C}})$  induced by the transformations  $\lambda^{\mathbb{Q}}$  and  $\lambda^{\mathbb{C}}$  of  $Ch^+ \mathbb{Q}$  and  $Ch^+ \mathbb{C}$  respectively.

**Transformation  $\mu$ :** similarly, the transformation  $\mu_K^{\mathcal{H}dg} : s_{\mathcal{H}dg} \Delta s_{\mathcal{H}dg} \rightarrow s_{\mathcal{H}dg} D$  is  $\mu^{\mathcal{H}dg} = (\mu_{K_{\mathbb{Q}}}^{\mathbb{Q}}, \mu_{K_{\mathbb{C}}}^{\mathbb{C}}, \mu_{\widetilde{K}}^{\mathbb{C}})$  where  $\mu_{\mathbb{Q}}$ ,  $\mu_{\mathbb{C}}$  and  $\widetilde{\mu}$  are the usual natural transformations of  $Ch^+ \mathbb{Q}$  and  $Ch^+ \mathbb{C}$  respectively.

**THEOREM 6.5.28.** *The category  $(\mathcal{H}dg, s_{\mathcal{H}dg}, E_{\mathcal{H}dg}, \mu_{\mathcal{H}dg}, \lambda_{\mathcal{H}dg})$  is an additive cosimplicial descent category.*

*In addition, the forgetful functor  $U : \mathcal{H}dg \rightarrow Ch^+ \mathbb{Q}$  given by  $U((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) = K_{\mathbb{Q}}$  is a functor of additive cosimplicial descent categories.*

By proposition 6.4.27, the simple functor  $(s, \delta) : \Delta CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$  preserves  $E_2$ -isomorphisms. In addition, it also preserves filtered quasi-isomorphisms, [DeII]7.I.6.2.

**LEMMA 6.5.29.** *If  $\mathcal{A}$  is an abelian category, the functor  $(s, \delta) : \Delta CF^+ \mathcal{A} \rightarrow CF^+ \mathcal{A}$  preserves filtered quasi-isomorphisms.*

*That is to say, if  $f : (A, W) \rightarrow (B, V)$  is a morphism of cosimplicial filtered*

cochain complexes such that  $+f^m : (A^m, W) \rightarrow (B^m, V)$  is a filtered quasi-isomorphism for each  $m$ , then  $\mathbf{s}f : (\mathbf{s}A, \delta W) \rightarrow (\mathbf{s}B, \delta V)$  is also a filtered quasi-isomorphism.

**Proof of 6.5.28.**

We will apply the transfer lemma 2.5.8<sup>op</sup> to  $U : \mathcal{H}dg \rightarrow Ch^+\mathbb{Q}$ .

(SDC 1)<sup>op</sup> holds since  $\mathcal{H}dg$  is additive. Let us see (SDC 3)<sup>op</sup>, that is, let us check that  $\mathbf{s}_{\mathcal{H}dg} = (\mathbf{s}, \delta, \mathbf{s}) : \Delta\mathcal{H}dg \rightarrow \mathcal{H}dg$  is indeed a functor.

Given  $K \in \Delta\mathcal{H}dg$ , then  $\mathbf{s}_{\mathcal{H}dg}K$  is a Hodge complex by [DeIII] 8.I.15 i).

Hence,  $\mathbf{s}_{\mathcal{H}dg}(K)$  satisfies conditions a) and b) of definition 6.5.14. Indeed, they are consequences of the functoriality of  $(\mathbf{s}, \mathbf{s})$  and  $(\mathbf{s}, \delta)$ , and it can be proven that  $H^k(\mathbf{s}K_{\mathbb{Q}})$  is a finite dimensional vector space using the standard argument of the proof of (SDC 6) in proposition 5.2.1 (or equivalently, using the spectral sequence associated with  $\mathbf{s}K_{\mathbb{Q}}$ ).

On the other hand, by assumption  $\alpha = (\alpha_0, \alpha_1, (\tilde{K}, \tilde{W}))$  is such that  $\alpha_i$  is a degreewise filtered quasi-isomorphism for  $i = 0, 1$ . Then, from 6.5.29 we deduce that  $(\mathbf{s}, \delta)\alpha_i$  is so, for  $i = 0, 1$ . Therefore, the zig-zag  $\mathbf{s}\alpha$  given by formula (6.7) satisfies condition c) of the definition of mixed Hodge complex. Thus, it remains to see (MHC).

Given an integer  $n$ ,  $(\delta_W \mathbf{Gr}_n(\mathbf{s}K_{\mathbb{Q}}), (\delta_W \mathbf{Gr}_n(\mathbf{s}K_{\mathbb{C}}), \mathbf{s}F), \mathbf{Gr}_n(\mathbf{s}\alpha))$  satisfies the hypothesis of definition 6.5.5 of Hodge complex of weight  $n$  by loc. cit., except condition c) which is trivially satisfied because each  $\mathbf{s}\alpha_i$  is a filtered quasi-isomorphism.

Let us check now the functoriality of  $\mathbf{s}_{\mathcal{H}dg}$  with respect to the morphisms of  $\Delta\mathcal{H}dg$ .

A morphism  $f = (f_{\mathbb{Q}}, f_{\mathbb{C}}, \tilde{f}) : ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha) \rightarrow ((K'_{\mathbb{Q}}, W'), (K'_{\mathbb{C}}, W', F'), \alpha')$  in  $\Delta\mathcal{H}dg$  gives rise to the following commutative diagram of  $\Delta CF^+\mathbb{C}$

$$\begin{array}{ccccc} (K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \\ f_{\mathbb{C}} \downarrow & & \tilde{f} \downarrow & & \downarrow f_{\mathbb{Q}} \otimes \mathbb{C} \\ (K'_{\mathbb{C}}, W') & \xleftarrow{\alpha'_0} & (\tilde{K}', \tilde{W}') & \xrightarrow{\alpha'_1} & (K'_{\mathbb{Q}}, W') \otimes \mathbb{C} . \end{array}$$

Therefore, applying  $(\mathbf{s}, \delta) : \Delta CF^+\mathbb{C} \rightarrow CF^+\mathbb{C}$  we get a commutative diagram



in  $\text{CF}^+\mathbb{C}$ , that gives rise to

$$\begin{array}{ccccccc}
(\mathbf{s}K_{\mathbb{C}}, \delta W) & \xleftarrow{\mathbf{s}\alpha_0} & (\mathbf{s}\tilde{K}, \delta\tilde{W}) & \xrightarrow{\mathbf{s}\alpha_1} & (\mathbf{s}(K_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W \otimes \mathbb{C})) & \xrightarrow{\sim} & (\mathbf{s}K_{\mathbb{Q}}, \delta W) \otimes \mathbb{C} \\
\downarrow \mathbf{s}f_{\mathbb{C}} & & \downarrow \tilde{f} & & \downarrow \mathbf{s}(f_{\mathbb{Q}} \otimes \mathbb{C}) & & \downarrow (\mathbf{s}f_{\mathbb{Q}}) \otimes \mathbb{C} \\
(\mathbf{s}K'_{\mathbb{C}}, \delta W') & \xleftarrow{\mathbf{s}\alpha'_0} & (\mathbf{s}\tilde{K}', \delta\tilde{W}') & \xrightarrow{\mathbf{s}\alpha'_1} & (\mathbf{s}(K'_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W' \otimes \mathbb{C})) & \xrightarrow{\sim} & (\mathbf{s}K'_{\mathbb{Q}}, \delta W') \otimes \mathbb{C} .
\end{array}$$

Therefore,  $\mathbf{s}_{\mathcal{H}dg}f = (\mathbf{s}f_{\mathbb{Q}}, \mathbf{s}f_{\mathbb{C}}, \mathbf{s}\tilde{f})$  is a morphism in  $\mathcal{H}dg$ .

Now we will prove (SDC 4)<sup>'op</sup> and (SDC 5)<sup>'op</sup>. Denote by  $\lambda^{\mathbb{Q}}, \lambda^{\mathbb{C}}$  the natural transformations relative the descent categories  $\text{CF}^+\mathbb{Q}$  and  $\text{CF}^+\mathbb{C}$  (with the structure given in 6.4.27). These transformations coincide at the level of cochain complexes with the usual transformation  $\lambda$  of 6.1.

If  $((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \alpha)$  is a mixed Hodge complex, from 6.4.27 and 6.4.7, it follows that  $\lambda^{\mathbb{Q}}_{K_{\mathbb{Q}}}, \lambda^{\mathbb{C}}_{K_{\mathbb{C}}}$  and  $\lambda^{\mathbb{C}}_{\tilde{K}}$  preserve the filtrations. Set  $L = L \times \Delta$ . We state that the following diagram commutes in  $\text{CF}^+\mathbb{C}$

$$\begin{array}{ccccccc}
(K_{\mathbb{C}}, W) & \xleftarrow{\alpha_0} & (\tilde{K}, \tilde{W}) & \xrightarrow{\alpha_1} & (K_{\mathbb{Q}}, W) \otimes \mathbb{C} & & \\
\downarrow \lambda^{\mathbb{C}}_{K_{\mathbb{C}}} & & \downarrow \lambda^{\mathbb{C}}_{\tilde{K}} & & \downarrow \lambda^{\mathbb{C}}_{K_{\mathbb{Q}} \otimes \mathbb{C}} & \searrow \lambda^{\mathbb{Q}}_{K_{\mathbb{Q}}} \otimes \mathbb{C} & \\
(\mathbf{s}(K_{\mathbb{C}}), \delta(W)) & \xleftarrow{\mathbf{s}(\alpha_0)} & (\mathbf{s}(\tilde{K}'), \delta(\tilde{W})) & \xrightarrow{\mathbf{s}(\alpha_1)} & (\mathbf{s}(K'_{\mathbb{Q}} \otimes \mathbb{C}), \delta(W' \otimes \mathbb{C})) & \xrightarrow{\sim} & (\mathbf{s}(K'_{\mathbb{Q}}), \delta(W')) \otimes \mathbb{C} .
\end{array}$$

Indeed, the squares commutes by the functoriality of  $\lambda^{\mathbb{C}}$ , as well as the right triangle since  $\lambda_L$  is just the inclusion of  $L$  as direct summand of  $\mathbf{s}(L \times \Delta)$ .

Consequently  $\lambda_{\mathcal{H}dg} = (\lambda^{\mathbb{Q}}_{K_{\mathbb{Q}}}, \lambda^{\mathbb{C}}_{K_{\mathbb{C}}}, \lambda^{\mathbb{C}}_{\tilde{K}})$  is a morphism in  $\mathcal{H}dg$ . It can be argued similarly with  $\mu^{\mathcal{H}dg}_K = (\mu^{\mathbb{Q}}_{K_{\mathbb{Q}}}, \mu^{\mathbb{C}}_{K_{\mathbb{C}}}, \mu^{\mathbb{C}}_{\tilde{K}})$ .

(FD 1)<sup>op</sup> is trivial since  $\mathbf{U}$  is additive, and the diagram

$$\begin{array}{ccc}
\Delta\mathcal{H}dg & \xrightarrow{\Delta\mathbf{U}} & \Delta Ch^+\mathbb{Q} \\
\downarrow (\mathbf{s}, \delta, \mathbf{s}) & & \downarrow \mathbf{s} \\
\mathcal{H}dg & \xrightarrow{\mathbf{U}} & Ch^+\mathbb{Q}
\end{array}$$

commutes. Finally, it is clear that the transformations  $\lambda$  of  $\mathcal{H}dg$  and  $Ch^+\mathbb{Q}$  are compatible, and the same happens with  $\mu$ , so 2.5.8<sup>op</sup> holds.  $\square$

**REMARK 6.5.30.** The fixed length of the zig-zag  $\alpha$  of a mixed Hodge complex has no relevance in the previous proof. Consequently, the category

$$(\mathcal{H}dg_{\mathcal{G}}, \mathbf{s}_{\mathcal{H}dg}, E_{\mathcal{H}dg}, \mu_{\mathcal{H}dg}, \lambda_{\mathcal{H}dg})$$

defined similarly, is an additive cosimplicial descent category, and the forgetful functor  $U : \mathcal{H}dg \rightarrow Ch^+\mathbb{Q}$  is again a functor of additive cosimplicial descent categories.

## Appendix A

# Eilenberg-Zilber-Cartier Theorem

We will need the following theorem, known as the Eilenberg-Zilber-Cartier ([DP],2.9).

**THEOREM A.1.1** (Eilenberg-Zilber-Cartier).

*Consider an additive category  $\mathcal{U}$ , and the square*

$$\begin{array}{ccc} \Delta^\circ \Delta^\circ \mathcal{U} & \xrightarrow{\Delta^\circ K} & \Delta^\circ Ch_+ \mathcal{U} \\ \downarrow D & & \downarrow K \\ & & Ch_+ Ch_+ \mathcal{U} \\ & & \downarrow Tot \\ \Delta^\circ \mathcal{U} & \xrightarrow{K} & Ch_+ \mathcal{U}. \end{array}$$

a) If  $V \in \Delta^\circ \Delta^\circ \mathcal{U}$ , then the morphisms  $Id_{V_{0,0}} : [Tot K \Delta^\circ K(V)]_0 \rightarrow [KD(V)]_0$  and  $Id_{V_{0,0}} : [KD(V)]_0 \rightarrow [Tot K \Delta^\circ K(V)]_0$  can be extended to universal morphisms

$$\begin{cases} \eta_{E-Z}(V) : Tot K \Delta^\circ K(V) \rightarrow KD(V) \\ \mu_{E-Z}(V) : KD(V) \rightarrow Tot K \Delta^\circ K(V) \end{cases},$$

that are homotopy inverse.

b) Each universal morphism  $F : Tot K \Delta^\circ K(V) \rightarrow KD(V)$  with  $F_0 = Id$  is (universally) homotopic to  $\eta_{E-Z}(V)$ . Analogously, if  $G : KD(V) \rightarrow Tot K \Delta^\circ K(V)$  is universal and  $G_0 = Id$ , then  $G$  is (universally) homotopic to  $\mu_{E-Z}(V)$ .

**REMARK A.1.2.**

1) Given  $V \in \Delta^\circ \Delta^\circ \mathcal{U}$ , a morphism  $\oplus V_{p,q} \rightarrow \oplus V_{p',q'}$  in  $\mathcal{U}$  is universal if each

component  $V_{p,q} \rightarrow V_{p',q'}$  is of the form  $\sum_{\alpha,\beta} n_{\alpha,\beta} V(\alpha, \beta)$ , where  $n_{\alpha,\beta} \in \mathbb{Z}$ , and  $\alpha : [p'] \rightarrow [p]$  and  $\beta : [q'] \rightarrow [q]$  are morphisms of  $\Delta$ .

Similarly, a morphism  $F$  in  $Ch_+(\mathcal{U})$  between  $TotK\Delta^\circ K(V)$  and  $KD(V)$  is universal if each  $F_n$  is so.

II) Since  $\eta_{E-Z}$  and  $\mu_{E-Z}$  are universal, it follows that they are functorial in  $V$ , so they define natural transformations between  $KD$  and  $TotK\Delta^\circ K$ .

**Proof.** Given  $p, q, r, s \geq 0$ , let  $M(p, q; r, s)$  be the free abelian group generated by the pairs  $(\alpha, \beta)$ , where  $\alpha : [r] \rightarrow [p]$  and  $\beta : [s] \rightarrow [q]$  are morphisms in  $\Delta$ .

Consider the category  $\mathcal{M}$  whose objects are the symbols  $M_{p,q}$ ,  $p, q \geq 0$ . A morphism from  $M_{p,q}$  to  $M_{r,s}$  is just an element of  $M(p, q; r, s)$ . Composition in  $\mathcal{M}$  is inherited from composition in  $\Delta$ . Hence  $(\alpha', \beta')(\alpha, \beta) = (\alpha\alpha', \beta\beta')$  if  $(\alpha, \beta) \in M(p, q; r, s)$  and  $(\alpha', \beta') \in M(r, s; t, u)$ . Consequently,  $\Delta^\circ \times \Delta^\circ \subseteq \mathcal{M}$ .

Attach objects and morphisms to  $\mathcal{M}$  in such a way that we can consider finite direct sums in  $\mathcal{M}$ . In this way we get the additive category  $\widetilde{\mathcal{M}}$ , and again  $\Delta^\circ \times \Delta^\circ \subseteq \widetilde{\mathcal{M}}$ . Therefore, restricting the identity  $\widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}$  we obtain a simplicial object  $M \in \Delta^\circ \Delta^\circ \widetilde{\mathcal{M}}$ .

Consider  $V \in \Delta^\circ \Delta^\circ \mathcal{U}$ . Then  $V$  can be extended in a unique way to an additive functor  $\mathcal{M} \rightarrow \mathcal{U}$ , that gives rise to  $V' : Ch_+(\widetilde{\mathcal{M}}) \rightarrow Ch_+(\mathcal{U})$ .

Thus, a morphism in  $Ch_+(\mathcal{U})$  between  $TotK\Delta^\circ K(V)$  and  $KD(V)$  is universal if and only if it is the image under  $V'$  of a morphism of  $Ch_+(\widetilde{\mathcal{M}})$  between  $TotK\Delta^\circ K(M)$  and  $KD(M)$ . Moreover, since  $V : \widetilde{\mathcal{M}} \rightarrow \mathcal{U}$  is additive, a homotopy between two morphisms  $F$  and  $G$  from  $TotK\Delta^\circ K(M)$  to  $KD(M)$  (or viceversa) is mapped by  $V'$  into a (universal) homotopy between  $V'(F)$  and  $V'(G)$ , that are morphisms from  $TotK\Delta^\circ K(V)$  to  $KD(V)$  (or viceversa). Hence, we can restrict ourselves to  $\mathcal{U} = \widetilde{\mathcal{M}}$  and  $V = M$ .

Let  $Ab$  be the category of abelian groups. For each  $l \geq 0$  denote by  $K(l) \in \Delta^\circ Ab$  the simplicial abelian group such that  $K(l)_p$  is the free abelian group generated by the morphisms  $\alpha : [p] \rightarrow [l]$ . In other words,  $K(l)$  is obtained from the “standard” simplicial object  $\Delta[l]$ , by taking free groups. Consider  $K(l, m) \in \Delta^\circ \Delta^\circ Ab$  given by  $K(l, m)_{p,q} = K(l)_p \otimes K(m)_q$ .

Consider  $R_{p,q} : \Delta^\circ \Delta^\circ Ab \rightarrow Ab$  with  $R_{p,q}(W) = W_{p,q}$  if  $W \in \Delta^\circ \Delta^\circ Ab$ , and the natural transformations  $\tau : R_{p,q} \rightarrow R_{p',q'}$  (also called  $FD$ -operators). Denote by  $N(p, q; p', q')$  the group consisting of all of them. Examples of such  $\tau$  are the basic transformations  $(\alpha, \beta)^*(W) = W_{\alpha,\beta}$  if  $\alpha : [p'] \rightarrow [p]$  and  $\beta : [q'] \rightarrow [q]$ .

By [EM] 3.1 we have that  $N(p, q; p', q')$  is a free group generated by the basic transformations. In addition,  $\tau \in N(p, q; p', q')$  is characterized by its value at the bisimplicial abelian groups  $K(l, m)$ ,  $\forall l, m$ .

Clearly, the mapping  $(\alpha, \beta) \rightarrow (\alpha, \beta)^*$  is injective, since given any  $\alpha : [p'] \rightarrow [p]$  and  $\beta : [q'] \rightarrow [q]$  then  $(\alpha, \beta)^*(K(p, q))(Id_{[p]}, Id_{[q]}) = (\alpha, \beta) \in K(p, q)_{p', q'}$ . It follows that  $N(p, q; p', q') \simeq M(p, q; p', q')$ .

Therefore,  $\mathcal{M}$  can be replaced by the category  $\mathcal{N}$  whose objects are symbols  $N_{p,q}$  and whose morphisms between  $N_{p,q}$  and  $N_{p',q'}$  are just  $N(p, q; p', q')$ . Similarly, consider the restrictions of the identity functor  $\tilde{\mathcal{N}}$  and  $N \in \Delta^\circ \Delta^\circ \tilde{\mathcal{N}}$ . The morphism  $Id : N_{0,0} \rightarrow N_{0,0}$  in  $N(0, 0; 0, 0)$  is the natural transformation defined by  $Id : K(l, m)_{0,0} \rightarrow K(l, m)_{0,0}$  for all  $l, m$ .

By the classical Eilenberg-Zilber theorem [May] 29.3 it follows the existence of natural transformations defined from  $\eta(l, m) : \bigoplus_{p+q=n} K(l)_p \otimes K_q \rightarrow K(l)_n \otimes K(m)_n$  and  $\mu(l, m) : K(l)_n \otimes K(m)_n \rightarrow \bigoplus_{p+q=n} K(l)_p \otimes K_q$  for all  $l, m$ , such that they are the identity in degree 0, and such that they are homotopy inverse. In addition, any two such  $\eta$  are homotopic in a natural way, and the same holds for  $\mu$ .

By definition of  $\tilde{\mathcal{N}}$ , the morphisms  $\{\eta(l, m)\}_{l,m}$  correspond to morphisms  $\eta \in \bigoplus_{p+q=n} N(p, q; n, n)$ . In other words  $\eta : TotK\Delta^\circ K(N) \rightarrow KD(N)$ , and similarly  $\{\mu(l, m)\}_{l,m}$  correspond to  $\mu : KD(N) \rightarrow TotK\Delta^\circ K(N)$ , in such a way that (natural) homotopies are preserved by this correspondence. Hence the proof is finished.  $\square$

**REMARK A.1.3** (Description of  $\eta_{E-Z}$  and  $\mu_{E-Z}$ ).

Given  $V \in Ch_+Ch_+\mathcal{U}$ , the “shuffle” map  $\eta_{E-Z}(V) : TotK\Delta^\circ K(V) \rightarrow KD(V)$  and the Alexander-Whitney map  $\mu_{E-Z}(V) : KD(V) \rightarrow TotK\Delta^\circ K(V)$  are (universal) inverse homotopy equivalence ([DP] 2.15). They are given by  $\eta_{E-Z}(V) = \Sigma \eta_{i,j}(V) : \bigoplus_{i+j=k} V_{ij} \rightarrow V_{k,k}$

$$\eta_{i,j}(V) = \Sigma_{(\alpha,\beta)} sign(\alpha, \beta) V(\sigma^{\alpha_j} \sigma^{\alpha_{j-1}} \dots \sigma^{\alpha_1}, \sigma^{\beta_i} \sigma^{\beta_{i-1}} \dots \sigma^{\beta_1})$$

where the sum is indexed over the  $(i, j)$ -“shuffles”  $(\alpha, \beta)$  and  $sign(\alpha, \beta)$  denotes the sign of  $(\alpha, \beta)$  (see [EM]).

On the other hand  $\mu_{E-Z}(V) = \Sigma \mu_{i,j}(V) : V_{k,k} \rightarrow \bigoplus_{i+j=k} V_{i,j}$ , where

$$\mu_{i,j}(V) = V(d^0 \overset{k-i}{\dots} d^0, d^k d^{k-1} \dots d^{j+1}) .$$

**REMARK A.1.4.** The “shuffle”  $\tilde{\eta}_{E-Z}$  and Alexander-Whitney  $\tilde{\mu}_{E-Z}$  maps given in [DP] are not exactly those used in these notes.

The reason is that the total functor used in [DP],  $\widetilde{Tot} : Ch_+Ch_+\mathcal{U} \rightarrow Ch_+\mathcal{U}$ , is isomorphic but not the same used here (see 5.1.3). Indeed, given  $\{V^{i,j} ; d^i, d^j\} \in Ch_+Ch_+\mathcal{U}$ , then  $\widetilde{Tot}(V)$  has as boundary map  $\sum d^i + (-1)^i d^j$ .

Therefore,  $\widetilde{\eta}_{E-Z} : \widetilde{Tot}K\Delta^\circ K \rightarrow KD$  and  $\widetilde{\mu}_{E-Z} : KD \rightarrow \widetilde{Tot}K\Delta^\circ K$ .

Denote by  $\Gamma : Ch_+Ch_+\mathcal{U} \rightarrow Ch_+Ch_+\mathcal{U}$  and  $\Gamma : \Delta^\circ\Delta^\circ\mathcal{U} \rightarrow \Delta^\circ\mathcal{U}$  the functors that interchange the indexes in a double complex and in a bisimplicial object respectively. Then

$$\eta_{E-Z}(V) = \widetilde{\eta}_{E-Z}(\Gamma V) \text{ and } \mu_{E-Z}(V) = \widetilde{\mu}_{E-Z}(\Gamma V) .$$

Note that  $\widetilde{Tot}\Gamma = Tot$ ,  $D\Gamma = D$  and  $K\Delta^\circ K\Gamma = \Gamma K\Delta^\circ K$ . Hence,  $\widetilde{\mu}_{E-Z}(\Gamma V) : KD(\Gamma V) = KD(V) \rightarrow \widetilde{Tot}K\Delta^\circ K(\Gamma V) = TotK\Delta^\circ K$ , and similarly for  $\eta_{E-Z}$ .

We will use in these notes the Alexander-Whitney map in order to proof the factorization axiom in the (co)chain complexes case. We will need as well the following property of  $\mu_{E-Z}$ .

**PROPOSITION A.1.5.** *The natural transformation  $\mu_{E-Z}$  is associative. More concretely, given a trisimplicial object  $T$  in  $\mathcal{U}$ , the morphisms  $T_{n,n,n} \rightarrow \bigoplus_{r+s+t=n} T_{r,s,t}$  obtained by applying twice  $\mu_{E-Z}$*

$$\begin{array}{ccccc} & & \bigoplus_{i+j=n} T_{i,j,n} & & \\ & \nearrow^{\mu_{EZ}} & & \searrow^{\mu_{EZ}} & \\ T_{n,n,n} & & & & \bigoplus_{r+s+t=n} T_{r,s,t} \\ & \searrow_{\mu_{EZ}} & & \nearrow_{\mu_{EZ}} & \\ & & \bigoplus_{p+q=n} T_{n,p,q} & & \end{array}$$

*coincide.*

This is a well-known property (see, for instance, [H] 14.2.3.). It holds that  $\eta_{E-Z}$  is also associative, and in addition it is symmetric (that is, it is invariant by swapping the indexes). On the other hand,  $\mu_{E-Z}$  is not symmetric, but it is symmetric up to homotopy equivalence.

**REMARK A.1.6.** Usually the transformations  $\mu_{E-Z}$  and  $\eta_{E-Z}$  are used when  $\mathcal{U}$  is a category of  $R$ -modules, and the bisimplicial object considered is of the form  $\{X_n \otimes Y_m\}_{n,m}$  (for instance in the study of the relationship between the homology of the cartesian product of topological spaces and the tensorial product of their homologies).

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